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Exact operator bosonization of finite number of fermions in one space dimension

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Abstract: We derive an exact operator bosonization of a finite number of fermions in one space dimension. The fermions can be interacting or noninteracting and can have an arbitrary hamiltonian, as long as there is a countable basis of states in the Hilbert space. In the bosonized theory the finiteness of the number of fermions appears as an ultraviolet cut-off. We discuss implications of this for the bosonized theory. We also discuss applications of our bosonization to one-dimensional fermion systems dual to (sectors of) string theory such as LLM geometries and $c = 1$ matrix model.

Keywords: AdS-CFT Correspondence, Non-Commutative Geometry, D-branes.

Contents

1. Introduction

The recent study by Lin, Lunin and Maldacena [1] of a class of half-BPS type IIB geometries in asymptotically $AdS_5 \times S^5$ spaces, offers an excellent laboratory to explore aspects of quantum gravity. In the boundary super Yang-Mills theory, the corresponding half-BPS states are described by N free fermions in a harmonic potential [2–4]. At large N, there is a semiclassical description of the states of this system in terms of droplets of fermi fluid in phase space; LLM showed that there is a similar structure in the classical geometries in the bulk. The semiclassical correspondence is already remarkable in the sense that it exhibits a noncommutative structure of two of the space coordinates [1, 5]; however, finite N effects, corresponding to fully quantum mechanical aspects of bulk gravity, open up more interesting questions [6]. While it has been shown that semiclassically small fluctuations of the droplet boundaries correspond to small gravitational fluctuations around the classical geometries [1, 7, 8], at finite N only those fluctuations of the fermi system which have low enough excitation energy compared to N can be identified with gravity modes propagating in the bulk (we will elaborate on this in section 4.2.1). Excitations with energy comparable to or higher than N do not correspond to gravitons but to non-local objects in the bulk, namely giant gravitons or dual giant gravitons [9–11]. Even more remarkable is the fact that the fermi partition function can be mapped onto the partition function for giant gravitons or dual giant gravitons [12] alone, without involving any low-energy gravitational degrees of freedom at all. This seems to suggest that, at least in the half-BPS sector, the bulk geometry has a nontrivial structure at a small enough length scale whose precise value depends on N, and that below this length scale gravitational phenomena in the bulk are described by degrees of freedom that are quite different from the degrees of freedom that characterize low-energy gravitational fluctuations. In such an interpretation, low-energy gravity modes would be "composites" of the microscopic degrees of freedom. This provides a motivation to look for an exact bosonization of the finite N fermi system, which should provide the "right" variables to describe bulk gravity consistent with such a structure.

Bosonization of a system of finite number of fermions is an interesting problem in its own right, with many potential applications in quantum field theory as well as in condensed matter theory. For this reason the problem has received attention for more than half a century now. Approximate solutions have long been found in case of free nonrelativistic fermions near the fermi level [13] where the fermion density turns out to be the spatial derivative of the bosonic field. This bosonization becomes exact when the fermions are relativistic and are infinite in number [14, 15]. In the case of free nonrelativistic fermions in an inverted harmonic oscillator potential in one space dimension, arising in the context of $c = 1$ matrix models, approximate bosonization in terms of fermion density gives rise to the massless boson of two-dimensional noncritical string theory (usually called the "tachyon") [16 – 19]. An exact bosonization of fermions in an arbitrary potential in one space dimension was found in [20, 21] in terms of the Wigner phase space density of the fermions, but the bosonic variable satisfies an infinite dimensional nonabelian algebra (W-infinity), instead of the Heisenberg algebra, and it satisfies a quadratic constraint; the approximate bosonization in terms of the position space density can be derived from it. Recently it has been noticed [2, 3, 22, 1, 12] that the spectrum, and consequently the partition function, of N nonrelativistic fermions in a harmonic oscillator potential (discussed in the previous paragraph) agrees with that of a system of free nonrelativistic bosons which are infinite in number but each of which moves in an equally spaced N-level system (similar to harmonic oscillator energy levels with an upper cutoff). Indeed the fermionic spectrum also agrees [12] with that of a second bosonic system with a finite number N of particles, each moving in a harmonic oscillator potential. These two bosonic spectra represent those of giant gravitons [9] and dual giant gravitons [10, 11] respectively.

In the present work, we derive an *exact operator equivalence* between a system of N fermions in one space dimension and two different bosonic systems satisfying the usual commutation relations (Heisenberg algebra). The two bosonic systems are reminiscent of giant gravitons and dual giant gravitons, but they appear in bosonization of fermions moving in any potential. Perhaps the most remarkable effect of having a finite number of fermions is that it gives rise to fuzziness in coordinate space in the bosonized theory. This fuzziness can be seen directly in the bosonized theory and arises because N provides a high energy cut-off on the basic bosonic degrees of freedom. It can also be seen by reformulating the bosonized theory as a theory on a lattice with spacing given by $1/N$. In the LLM context, what this means is that for finite N we have a direct derivation of the appearance of a short-distance cut-off in the bulk string/gravity theory, at least in the half-BPS sector. This is consistent with what we expect from stringy exclusion principle [23]. Earlier works on the appearance of graininess on the gravity side of AdS/CFT correspondence are [24 – 27].

This paper is organized as follows. In the next section, we first give the rules for our first bosonization which maps the system of N fermions to a system of bosons each of which can occupy a state in an N-dimensional Hilbert space $\mathcal{H}_N$. The finiteness of the number of fermions reappears in the bosonized theory as finite dimensionality of the single-particle Hilbert space. Consequences of this for the bosonized theory are discussed in section 3. In particular, we argue that the quantum phase space of the bosons is fuzzy and compact. Actually finite N is responsible for graininess even in coordinate space. This is seen more directly in a lattice formulation of the bosonized theory, with lattice spacing $1/N$, which is also discussed in this section. Section 3 also includes a detailed discussion of the bosonic phase space density. In the LLM context, the bosonic density as a function of the phase space has the appearance of a rugged circular "cake" with an approximately fixed diameter, confirming the interpretation of the bosons as giant gravitons. This is to be contrasted with the fermionic phase space density which looks like droplet configurations. Section 4 is devoted to a detailed discussion of this and other aspects of application of our bosonization in the LLM context. In particular, we argue that at finite N, the LLM gravitons must be fuzzy in an essential way. Section 5 describes the second bosonization. The essential difference with the first bosonization is that the number N now appears as an upper limit on the total number of bosons. Application of our bosonization to the $c = 1$ matrix model is briefly sketched in section 6. Some details of computations are given in appendices A and B. In appendix C we discuss an important byproduct of our bosonization, namely the bosonization of N fermions in a finite (K) level system and the resulting bosonic construction of representations of U(K), which is different from the well-known Schwinger representation.

2. The first bosonization

Let us first set up the notation. Consider a system of N fermions each of which can occupy a state in an infinite-dimensional Hilbert space $\mathcal{H}_f$. Suppose there is a countable basis of $\mathcal{H}_f : \{|m\rangle, m = 0, 1, \ldots, \infty\}$. For example, this could be the eigenbasis of a single-particle hamiltonian, $\hat{h}|m\rangle = \mathcal{E}(m)|m\rangle$, although other choices of basis would do equally well, as long as it is a countable basis. In the second quantized notation we introduce creation (annihilation) operators $\psi^\dagger_m$ (ψ_m) which create (destroy) particles in the state $|m\rangle$. These satisfy the anticommutation relations

$$\{\psi_m, \psi^\dagger_n\} = \delta_{mn} \tag{2.1}$$

The N-fermion states are given by (linear combinations of)

$$|f_1, \ldots, f_N\rangle = \psi^\dagger_{f_1}\psi^\dagger_{f_2}\cdots\psi^\dagger_{f_N}|0\rangle_F \tag{2.2}$$

where f_m are arbitrary integers satisfying $0 \le f_1 < f_2 < \cdots < f_N$, and $|0\rangle_F$ is the usual Fock vacuum annihilated by $\psi_m, m = 0, 1, \ldots, \infty$.

One can create any of the states $|f_1, \ldots, f_N\rangle$ from the state $|F_0\rangle \equiv |0, 1, \ldots, N-1\rangle$ by repeated application of operators

$$\Phi_{mn} = \psi^\dagger_m\psi_n \tag{2.3}$$

Properties of Φ_{mn} and related operators, including the Wigner and Husimi distributions, are listed in appendix B.

We will map the above fermionic system to a system of bosons each of which can occupy a state in an N-dimensional Hilbert space $\mathcal{H}_N$. Suppose we choose a basis $\{|i\rangle,\ i = 1, \ldots, N\}$ of $\mathcal{H}_N$. In the second quantized notation we introduce creation (annihilation) operators $a^\dagger_i$ (a_i) which create (destroy) particles in the state $|i\rangle$. These satisfy the commutation relations

$$[a_i, a^\dagger_j] = \delta_{ij}, \qquad i, j = 1, \ldots, N \tag{2.4}$$

A state of this bosonic system is given by (a linear combination of)

$$|r_1, \ldots, r_N\rangle = \frac{(a^\dagger_1)^{r_1}\cdots(a^\dagger_N)^{r_N}}{\sqrt{r_1!\cdots r_N!}}|0\rangle_B \tag{2.5}$$

2.1 The bosonization formulae

The bosonization formulae are written most economically using the notions of operator delta and theta-functions. These are defined for any operator $\hat{O}$ as follows:

$$\delta(\hat{O}) \equiv \int_0^{2\pi} \frac{d\theta}{2\pi} \, \exp(i\theta\hat{O}), \tag{2.6}$$

and

$$\theta_+(\hat{O}) \equiv \sum_{m=0}^{\infty} \delta(\hat{O} - m), \qquad \theta_-(\hat{O}) \equiv 1 - \theta_+(\hat{O}). \tag{2.7}$$

Furthermore, we will also need to introduce the following operators:

$$\sigma_k \equiv \frac{1}{\sqrt{a_k^\dagger a_k + 1}} a_k, \qquad \sigma_k^\dagger \equiv a_k^\dagger \frac{1}{\sqrt{a_k^\dagger a_k + 1}} \tag{2.8}$$

These operators have interesting properties. In particular, the following relations are useful in obtaining the bosonization formulae given below.

$$\sigma_k \, \sigma_k^\dagger = 1, \qquad \sigma_k^\dagger \, \sigma_k = \theta_+(a_k^\dagger a_k - 1). \tag{2.9}$$

Since delta-function operators of the form $\delta(\sum_{i=1}^n a_{k_i}^\dagger a_{k_i} - p)$ will appear quite often in the formulae given below, we give an alternative, more familiar, expression for it in terms of more elementary operators. First recall the following representation of the operator $|m\rangle\langle n|$ in a harmonic oscillator Hilbert space with raising (lowering) operators $a^\dagger(a)$:

$$|m\rangle\langle n| = \frac{1}{\sqrt{m!n!}} : a^{\dagger\, m} e^{-a^\dagger a} a^n : \tag{2.10}$$

where $::$ represents normal ordering. This implies

$$\delta(a^\dagger a - n) = |n\rangle\langle n| = \frac{1}{n!} : a^{\dagger\, n} e^{-a^\dagger a} a^n : \tag{2.11}$$

which can be used to write

$$\begin{aligned} \delta(\sum_{i=1}^n a_{k_i}^\dagger a_{k_i} - p) &= \sum_{r_{k_1}+\cdots+r_{k_n}=p} \prod_{i=1}^n \delta(a_{k_i}^\dagger a_{k_i} - r_{k_i}) \\ &= \sum_{r_{k_1}+\cdots+r_{k_n}=p} \frac{1}{r_{k_1}! \cdots r_{k_n}!} : a_{k_1}^{\dagger\, r_{k_1}} \cdots a_{k_n}^{\dagger\, r_{k_n}} \, e^{-\sum_{i=1}^n a_{k_i}^\dagger a_{k_i}} a_{k_1}^{r_{k_1}} \cdots a_{k_n}^{r_{k_n}} : \end{aligned} \tag{2.12}$$

Similarly, one can give alternative definitions of the operators $\delta(\sum_{m=m_1}^{m_2} \psi_m^\dagger \psi_m)$ which appear extensively in eqs. (2.17), (2.18) below. We first note the identities

$$\delta(\psi_n^\dagger \psi_n) = 1 - \psi_n^\dagger \psi_n, \quad \delta(\psi_n^\dagger \psi_n - 1) = \psi_n^\dagger \psi_n. \tag{2.13}$$

These identities enable us to write the following alternative expressions for some fermionic delta-functions:

$$\delta(\sum_{m=m_1}^{m_2} \psi_m^\dagger \psi_m) = \prod_{m=m_1}^{m_2} \delta(\psi_m^\dagger \psi_m) = \prod_{m=m_1}^{m_2} (1 - \psi_m^\dagger \psi_m). \tag{2.14}$$

We are now ready to describe the bosonization formulae.

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2.1.1 Fermionic representation of bosonic oscillators

We will first define the bosonic creation and annihilation operators in terms of their action on the states of the fermion system. We have, on a general N-fermion state (2.2),

$$\begin{aligned} a_k^\dagger \,|f_1,\ldots,f_N\rangle &= \sqrt{f_{N-k+1}-f_{N-k}}\,|f_1,\ldots,f_{N-k},f_{N-k+1}+1,\ldots,f_N+1\rangle, \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad k=1,\ldots,N-1 \\ a_N^\dagger \,|f_1,\ldots,f_N\rangle &= \sqrt{f_1+1}\,|f_1+1,\ldots,f_N+1\rangle. \end{aligned} \tag{2.15}$$

Thus, $a_k^\dagger$ moves each of the top k fermions, counting down from the topmost filled level, up by one step. Similarly, the action of a_k is to move each of the top k fermions down by one step:

$$\begin{aligned} a_k \,|f_1,\ldots,f_N\rangle &= \sqrt{f_{N-k+1}-f_{N-k}-1}\,|f_1,\ldots,f_{N-k},f_{N-k+1}-1,\ldots,f_N-1\rangle, \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad k=1,\ldots,N-1 \\ a_N \,|f_1,\ldots,f_N\rangle &= \sqrt{f_1}\,|f_1-1,\ldots,f_N-1\rangle. \end{aligned} \tag{2.16}$$

The reasoning that led us to these expressions for the bosonic creation and annihilation operators has been explained in appendix A.1. Here we simply mention that these definitions of the bosonic creation and annihilation operators satisfy the oscillator algebra (2.4). The reader can find a proof of this in appendix A.2. Note that the state $|F_0\rangle = |0,\,1,\,\ldots,\,N-1\rangle$ is special since it is annihilated by all the annihilation operators $a_k,\ k=1,\,2,\,\ldots,\,N$. Therefore, as expected, $|F_0\rangle$ is the oscillator vacuum state $|0\rangle_B$.

One can also give operator expressions for the oscillator creation and annihilation operators in terms of the fermion bilinears. We have,

$$\begin{aligned} a_k^\dagger \equiv & \sum_{m_k>m_{k-1}>\cdots>m_0} \sqrt{m_1-m_0}\;(\psi^\dagger_{m_0}\psi_{m_0})(\psi^\dagger_{m_1+1}\psi_{m_1})\cdots(\psi^\dagger_{m_k+1}\psi_{m_k}) \\ & \times\delta\Big(\sum_{m=m_0+1}^{m_1-1}\psi^\dagger_m\psi_m\Big)\,\delta\Big(\sum_{m=m_1+1}^{m_2-1}\psi^\dagger_m\psi_m\Big)\cdots\,\delta\Big(\sum_{m=m_{k-1}+1}^{m_k-1}\psi^\dagger_m\psi_m\Big) \\ & \times\delta\Big(\sum_{m=m_k+1}^{\infty}\psi^\dagger_m\psi_m\Big), \qquad\qquad k=1,2,\ldots,(N-1) \end{aligned} \tag{2.17}$$

and

$$\begin{aligned} a_N^\dagger \equiv & \sum_{m_N>m_{N-1}>\cdots>m_1} \sqrt{m_1+1}\;(\psi^\dagger_{m_1+1}\psi_{m_1})\cdots(\psi^\dagger_{m_N+1}\psi_{m_N}) \\ & \times\delta\Big(\sum_{m=m_1+1}^{m_2-1}\psi^\dagger_m\psi_m\Big)\cdots\,\delta\Big(\sum_{m=m_{N-1}+1}^{m_N-1}\psi^\dagger_m\psi_m\Big) \\ & \times\delta\Big(\sum_{m=m_N+1}^{\infty}\psi^\dagger_m\psi_m\Big). \end{aligned} \tag{2.18}$$

The annihilation operators are obtained from these by conjugation. These expressions look complicated, but it is easy to see that when acting on a generic fermion state (2.2), because of the operator delta-functions, only that term in the sum survives for which $m_k = f_N$, $m_{k-1} = f_{N-1}, \ldots, m_0 = f_{N-k}$ in (2.17) and $m_N = f_N$, $m_{N-1} = f_{N-1}, \ldots, m_1 = f_1$ in (2.18). This reproduces (2.15). One can similarly reproduce (2.16) from the action of conjugates of (2.17) and (2.18) on the generic fermion state (2.2).

Note that the bosonic oscillators are mapped into combinations of fermion *bilinears*, (2.3). This is because we are interested in a fixed fermion number sector of the fermion Fock space.

2.1.2 Fermion bilinears $\psi_m^\dagger \psi_n$ in terms of bosonic oscillators

The inverse map gives fermions in terms of the bosonic oscillators. Since the total number of fermions is conserved, this bosonization map can only relate fermion bilinears to the bosonic operators. The generic fermion bilinear is $\psi_m^\dagger \psi_n$, where $m, n = 0, 1, \cdots, \infty$, but it is sufficient for us to obtain a bosonized expression for the bilinear for $m \geq n$ only. The bosonized expression for $m < n$ can be obtained from this by conjugation. Before going to the most general expression (see (A.8)) we will first describe the relatively simple expressions obtainable for (i) for small values of $(m-n)$ and arbitrary N, and (ii) small values of N and $(m-n)$ any positive integer. We list below expressions for a few examples of both kinds.

$N = 1$**:** This is the simplest case. Here there is only one creation (annihilation) operator, $a^\dagger$ (a). We have,

$$\begin{aligned} \psi_n^\dagger\ \psi_n &= \ \delta(a^\dagger a - n) \\ \psi_{n+m}^\dagger\ \psi_n &= \ \sigma^{\dagger^m}\ \delta(a^\dagger a - n) \end{aligned} \tag{2.19}$$

$N = 2$**:** In this case there are two creation (annihilation) operators, $a_1^\dagger$ (a_1) and $a_2^\dagger$ (a_2). The bosonized expressions are now more complicated than for $N = 1$ case, but still manageable. We have,

$$\begin{aligned} \psi_n^\dagger\ \psi_n &= \ \delta(a_1^\dagger a_1 + a_2^\dagger a_2 - n + 1) + \delta(a_2^\dagger a_2 - n) \\ \psi_{n+m}^\dagger\ \psi_n &= \ \sigma_1^{\dagger^m}\ \delta(a_1^\dagger a_1 + a_2^\dagger a_2 - n + 1) + \sigma_1^m\ \sigma_2^{\dagger^m}\ \theta_+(a_1^\dagger a_1 - m)\ \delta(a_2^\dagger a_2 - n) \\ &\quad - \sum_{r_1=0}^{m-2} \sigma_1^{\dagger^{m-2-r_1}}\ \sigma_1^{r_1}\ \sigma_2^{\dagger^{r_1+1}}\ \delta(a_1^\dagger a_1 - r_1)\ \delta(a_2^\dagger a_2 - n) \end{aligned} \tag{2.20}$$

Arbitrary N**:** In this case relatively simple expressions exist only for small values of $(m-n)$. We give below expressions for $m = n$, $n+1$ and $n+2$.

$$\begin{aligned} \psi_n^\dagger\ \psi_n &= \ \sum_{k=1}^{N} \delta\Big(\sum_{i=k}^{N} a_i^\dagger a_i - n + N - k\Big) \\ \psi_{n+1}^\dagger\ \psi_n &= \ \sigma_1^\dagger\ \delta\Big(\sum_{i=1}^{N} a_i^\dagger a_i - n + N - 1\Big) \\ &\quad + \sum_{k=1}^{N-1} \sigma_k\ \sigma_{k+1}^\dagger\ \theta_+(a_k^\dagger a_k - 1)\ \delta\Big(\sum_{i=k+1}^{N} a_i^\dagger a_i - n + N - k - 1\Big) \end{aligned}$$

$$
\begin{aligned}
\psi^\dagger_{n+2}\,\psi_n = \ & {\sigma_1^\dagger}^2\,\delta\Big(\sum_{i=1}^N a_i^\dagger a_i - n + N - 1\Big) \\
& + \sum_{k=1}^{N-1} \sigma_k^2\,{\sigma_{k+1}^\dagger}^2\,\theta_+(a_k^\dagger a_k - 2)\,\delta\Big(\sum_{i=k+1}^N a_i^\dagger a_i - n + N - k - 1\Big) \\
& - \sum_{k=2}^{N-1} \sigma_{k-1}\,\sigma_{k+1}^\dagger\,\theta_+(a_{k-1}^\dagger a_{k-1} - 1)\,\delta(a_k^\dagger a_k)\,\delta\Big(\sum_{i=k+1}^N a_i^\dagger a_i - n + N - k - 1\Big) \\
& - \sigma_2^\dagger\,\delta(a_1^\dagger a_1)\,\delta\Big(\sum_{i=1}^N a_i^\dagger a_i - n + N - 2\Big)
\end{aligned} \tag{2.21}
$$

Bosonized expression for the generic bilinear for arbitrary N is rather complicated and not particularly illuminating. It has therefore been relegated to the appendix and is given in (A.8).

To check the validity of the bosonization formulae, we need to check that the (W_∞) algebra (B.6), expressing the commutation relation of fermion bilinears, works out. The W_∞ algebra (B.6) works out for $N = 1,\ 2$ if one uses the bosonized expressions (2.19) and (2.20). The first case is somewhat trivial. However, the second case is nontrivial and the satisfaction of the algebra (B.6) requires delicate cancellations, as we have shown in appendix A. We have not yet completed a check of the algebra for general $m,\ n$ in the case of arbitrary N because of the complexity of the relevant formula, (A.8). However, because of the nontrivial way in which it checks out for $N = 2$, we are confident that it will also check out for arbitrary N. Moreover, as proved in appendix A, the algebra works out for arbitrary N for small values of $(m - n)$.

2.1.3 The bosonized hamiltonian

Let us now discuss the bozonization of the hamiltonian. Before discussing the most generic case (2.28), let us first ignore the fermion-fermion interactions. Let $\mathcal{E}(m),\ m = 0, 1, 2, \ldots$ be the exact single-particle spectrum of the fermions in that case ($\mathcal{E}(m)$ are the eigenvalues of the matrix E_{mn} of (2.28)). Then, the hamiltonian is given by

$$
H = \sum_{m=0}^{\infty} \mathcal{E}(m)\ \psi_m^\dagger \psi_m \tag{2.22}
$$

Using the bosonization formula (2.21), a bosonized expression for the hamiltonian can be easily worked out:

$$
\begin{aligned}
H &= \sum_{m=0}^{\infty} \mathcal{E}(m) \sum_{k=1}^{N} \delta\Big(\sum_{i=k}^N a_i^\dagger a_i - m + N - k\Big) \\
&= \sum_{k=1}^{N} \mathcal{E}\Big(N - k + \sum_{i=k}^N a_i^\dagger a_i\Big).
\end{aligned} \tag{2.23}
$$

The first equality above follows from the first line of (2.21). Notice that, in general, the hamiltonian will not be quadratic in the bosonic creation and annihilation operators.

Nevertheless, the bosonic states (2.5) diagonalize the hamiltonian because it only involves the occupation number operators.

For most potentials, the exact spectrum $\mathcal{E}(m)$ of the single-particle hamiltonian is unlikely to be known. However, what we need for the purposes of bosonization is any countable basis, which could be provided, for example, by a part of the single-particle hamiltonian $\hat{h}$ that is exactly diagonalizable. Thus, suppose, $\hat{h} = \hat{h}_0 + \hat{h}_1$ such that $|m\rangle,\ m = 0, 1, 2,\ \cdots$ is a countable eigenbasis of $\hat{h}_0$ with eigenvalues $\mathcal{E}_0(m)$. Note that we do not require $\hat{h}_1$ to be small compared to $\hat{h}_0$. Then, we have

$$\begin{aligned} H &= \sum_{m,n=0}^{\infty} \Big(\mathcal{E}_0(m)\delta_{mn} + \langle m|\hat{h}_1|n\rangle \Big)\ \psi_m^\dagger\ \psi_n \\ &= \sum_{k=1}^{N} \mathcal{E}_0\Big(N - k + \sum_{i=k}^{N} a_i^\dagger a_i \Big) + \sum_{m,n=0}^{\infty} \langle m|\hat{h}_1|n\rangle\ \psi_m^\dagger\ \psi_n. \end{aligned} \tag{2.24}$$

Given $\hat{h}_1$, the matrix elements $\langle m|\hat{h}_1|n\rangle$ can be easily calculated. Typically these matrix elements will not be diagonal [1]. Thus, to obtain the bosonized form of the hamiltonian, we will need to use not only the bosonized expression for the fermion bilinear in the first of (2.21), but also the expression (A.8) for the general bilinear $\psi_m^\dagger\,\psi_n$. It follows that in this basis the hamiltonian (2.24) is not automatically diagonal. The "law of conservation of difficulty" is operative here; the problem of finding the exact fermionic single-particle spectrum has reappeared as the problem of diagonalizing this bosonic hamiltonian!

Let us now consider a few special cases in some detail to illustrate how our bosonization works in practice.

Fermions in a harmonic potential. A drastic simplification occurs in this case since this potential gives rise to an equally spaced spectrum, namely $\mathcal{E}(m) = c_1\ m + c_2$. In this case, an exact expression for the bosonized hamiltonian can be worked out and it corresponds to a bunch of simple harmonic oscillators (see (2.23)):

$$H_{\text{equal}-\text{spacing}} = c_1 \sum_{k=1}^{N} k\ a_k^\dagger a_k + H_{\text{vac}}, \tag{2.25}$$

where $H_{\text{vac}} = \frac{c_1}{2}N(N-1) + c_2 N$ is the energy of the fermi ground state.

Free fermions on a circle. In this case the single-particle spectrum is given by $\mathcal{E}(n) = cn^2,\ c = 2\pi^2\hbar^2/mL^2$, where m is the mass of a fermion and L is the circumference of the circle. The novelty here is that except for the ground state, each of the levels is doubly degenerate. We will consider this case in some detail since it illustrates the generality of our bosonization. Moreover, this example is among the rare exactly solvable cases with a nonlinear spectrum.

[1]The allowed values of $m,\ n$ depend on the system under consideration. For example, let $\hat{h}_0$ be the harmonic oscillator hamiltonian and $\hat{h}_1$ a quartic anharmonic piece. In this case, the matrix element $\langle m|\hat{h}_1|n\rangle$ vanishes unless $|m-n| \leq 4$.

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The normalized single-particle eigenstates (assuming periodic boundary conditions) are $\chi_{\pm n}(x) = \frac{1}{\sqrt{L}}\, e^{\pm i2\pi nx/L}$, $n = 0, 1, 2, \ldots$ ($\chi_0(x) = \frac{1}{\sqrt{L}}$ is non-degenerate). The mode expansion of the fermion field will involve the corresponding annihilation (creation) operators $\tilde{\psi}_{\pm n}$ ($\tilde{\psi}^\dagger_{\pm n}$). To make contact with our bosonization, we introduce the following identifications: $\psi_{2n-1} = \tilde{\psi}_{+n}$ and $\psi_{2n} = \tilde{\psi}_{-n}$ for $n = 1, 2, \ldots$ (ψ_0 corresponds to the constant mode $\chi_0(x)$). This identification maps the two fermion modes corresponding to each of the degenerate single-particle levels to two consecutive modes in an auxiliary fermion problem which our bosonization technique can handle. Using this mapping we can now transcribe all the bosonization formulae to this case. In particular, the bosonized hamiltonian can be obtained as follows:

$$\begin{aligned} H_{\text{circle}} &= c\sum_{n=1}^{\infty} n^2 \left[\tilde{\psi}^\dagger_{+n}\,\tilde{\psi}_{+n} + \tilde{\psi}^\dagger_{-n}\,\tilde{\psi}_{-n}\right] \\ &= c\sum_{k=1}^{N}\sum_{n=1}^{\infty} n^2 \left[\psi^\dagger_{2n-1}\,\psi_{2n-1} + \psi^\dagger_{2n}\,\psi_{2n}\right] \\ &= c\sum_{k=1}^{N}\sum_{n=1}^{\infty} n^2 \left[\delta\Big(\sum_{i=k}^{N} a_i^\dagger a_i - 2n + 1 + N - k\Big) + \delta\Big(\sum_{i=k}^{N} a_i^\dagger a_i - 2n + N - k\Big)\right] \\ &= \frac{c}{4}\sum_{k=1}^{N}\left[N - k + \sum_{i=k}^{N} a_i^\dagger a_i + \frac{1}{2}\Big(1 - (-1)^{N-k+\sum_{i=k}^{N} a_i^\dagger a_i}\Big)\right]^2 \end{aligned} \tag{2.26}$$

The second equality above follows from our mapping of the degenerate levels to odd and even level of the auxiliary fermion problem and the third follows from the first of the bosonization formulae in (2.21). As an example, let us compute the energy of the vacuum state using the bosonized hamiltonian. On the vacuum state, the oscillator term vanishes. We get

$$H_{\text{circle}}\;|0\rangle_B = \frac{c}{4}\sum_{k=1}^{N}\left[N - k + \frac{1}{2}\Big(1 - (-1)^{N-k}\Big)\right]^2\;|0\rangle_B. \tag{2.27}$$

For N even, the eigenvalue becomes $2c[1^2 + 2^2 + \cdots + (N/2-1)^2] + c(N/2)^2$, while for N odd we get $2c[1^2 + 2^2 + \cdots + (\frac{N-1}{2})^2]$. These are precisely the correct energy eigenvalues of the N-fermion ground state in the two cases. One can similarly check that the bosonized hamiltonian in (2.26) correctly gives the energy eigenvalues for excited states.

For small fluctuations around the Fermi vacuum, one can define a semiclassical limit in which the standard relativistic boson emerges as low energy excitations. These excitations coincide with the ones created by the $a^\dagger$-oscillators. Details of this calculation will be presented elsewhere.

Notice that the fermionic hamiltonian (first line of (2.26)) is manifestly invariant under $n \to -n$, for any given n. It is reflected as degeneracies in the spectrum. This manifest symmetry of the hamiltonian is lost in the bosonized form (last line of (2.26)), although of course the bosonic spectrum does display the appropriate degeneracies. For example, consider N even. In this case the fermi ground state is doubly degenerate since the single fermion at the top can occupy either of the two degenerate states labeled by $n = \pm N/2$. In the bosonic language, this pair of degenerate states is $|0\rangle_B$ and $a_1^\dagger|0\rangle_B$. Similarly, for N odd, the fermi ground state is unique, but the first excited state has four-fold degeneracy. The corresponding four degenerate bosonic states are $a_1^\dagger|0\rangle_B,\ {a_1^\dagger}^2|0\rangle_B,\ a_2^\dagger|0\rangle_B$ and $a_2^\dagger\ a_1^\dagger|0\rangle_B$. There will be a maximum of 2^N degenerate states in the most general case. It is an important problem to understand in a more systematic way this realization of the symmetry structure in the bosonic theory. This would open up the possibility of applications of our bosonization to problems in higher than 1 space dimension. For example, in 3 space dimensions in a potential with spherical symmetry, one can proceed as above and give some assignment of degenerate angular momentum states to the auxiliary fermion problem [2]. The important issue would then be to understand how rotational symmetry is realized in the bosonized theory. A systematic analysis of this is clearly very important, but is beyond the scope of the present work.

2.1.4 Interacting fermion models

Our discussion so far has been confined to noninteracting fermions for which the full hamiltonian is a sum of single-particle hamiltonians. The generic many-fermion hamiltonian is of the form

$$H_F = \sum_{m,n} \frac{1}{2} E_{mn} \psi_m^\dagger \psi_n + \sum_{m,n,l,r} V_{mnlr} \psi_m^\dagger \psi_n \psi_l^\dagger \psi_r + \cdots \tag{2.28}$$

where $E_{mn} = E_{nm}^*$ and $V_{mnlr}, \ldots$ satisfy appropriate relations to ensure hermiticity of H_F . By using our bosonization formula (A.8) we can get the bosonic version of H_F in terms of the $a, a^\dagger$. (Similarly we can write down a second bosonized version, in terms of $b, b^\dagger$ using the results of section 5.) It would be interesting to work out the properties of these bosons for fermions with a Coulomb interaction, for example, and compare with the standard collective excitations.

3. General properties of the bosonized theory

Let us now explore some consequences of our bosonization. We will first comment on the finite dimensionality of the single-particle Hilbert space.

3.1 Single particle: quantum phase space is fuzzy and compact

A finite-dimensional single-particle Hilbert space $\mathcal{H}_N$ is equivalent to a noncommutative or fuzzy compact phase space (for reviews on fuzzy spaces, see, e.g. [28, 29]). E.g, if the N states $|i\rangle$ are the first N states of a simple harmonic oscillator:[3]

$$\begin{aligned} |i\rangle &= \frac{\alpha^{\dagger i}}{\sqrt{i!}}|0\rangle, \qquad i = 0, 1, 2, \ldots, N-1, \qquad \alpha|0\rangle = 0 \\ \hat{h} &= \hbar\left(\alpha^\dagger\alpha + \frac{1}{2}\right), \qquad [\alpha, \alpha^\dagger] = 1, \qquad \hat{h}|i\rangle = \left(i + \frac{1}{2}\right)\hbar|i\rangle \,. \end{aligned} \tag{3.1}$$

[2]One could choose an assignment by switching on a small magnetic field that breaks rotational symmetry. The original problem is recovered by letting the magnetic field go to zero after bosonization is done.

[3]We will use α, $\alpha^\dagger$ to denote the lowering/raising operators of the single particle Hilbert space of a harmonic oscillator (as against a_i, $a_i^\dagger$ which are particle creation and annihilation operators).

then the phase space corresponds to a fuzzy disc (see e.g. [30] and (3.4) below.) Similarly, if the N states are the $2j+1$ states $|j,m\rangle, m=-j,\dots,+j$ of a rotor then the phase space is a fuzzy sphere [31].

The fuzziness or noncommutativity of the quantum phase space $\mathcal{M}$ follows from the existence of a finite $\hbar$, irrespective of whether N is finite or infinite(the infinite-dimensional Hilbert space of a one-dimensional particle corresponds to a plane with noncommutative coordinates $[x,p]=i\hbar$). See section B for more.

The compactness of $\mathcal{M}$ follows from finite N. Intuitively, when the states $|i\rangle$ are, e.g., "energy levels" of a bounded hamiltonian, the N Bohr orbits occupy a finite area of the phase space. A more precise construction goes as follows [30]. Consider $\mathcal{H}_N$ as a subspace of an infinite dimensional Hilbert space $\mathcal{H}_\infty$. For definiteness we will consider the case of the harmonic oscillator, given by (3.1), the generalization to other cases being straightforward in principle. The quantum phase space can be defined in terms of the algebra $\mathcal{A}_N$ of operators on $\mathcal{H}_N$, which themselves can be defined from operators on $\mathcal{H}_\infty$ using a projection operator:

$$\begin{aligned}&\hat{O}\to\hat{O}_N\equiv P_N\ \hat{O}\ P_N=\langle i|\hat{O}|j\rangle|i\rangle\langle j|\in\mathcal{A}_N,\ i,j=1,\dots,N\\&P_N\equiv\sum_1^N|i\rangle\langle i|\end{aligned}\tag{3.2}$$

Of course, the map $\hat{O}\to\hat{O}_N$ is many-to-one. In terms of the phase space (Husimi) representation of operators (see appendix B), (3.2) reads

$$\mathcal{O}(z,\bar{z})\to\mathcal{O}_N(z,\bar{z})\equiv P_N(z,\bar{z})*\mathcal{O}(z,\bar{z})*P_N(z,\bar{z})\tag{3.3}$$

where the star product is the Voros star product. The operation on the r.h.s. essentially turns the support of $\mathcal{O}(z,\bar{z})$ into a compact one of an approximate size

$$r_0^2=N\hbar\tag{3.4}$$

with an exponential tail $\sim\exp[-r^2/\hbar]$. This can be seen by noting that $P_N(z,\bar{z})=\Gamma(N+1,\frac{r^2}{2\hbar})/\Gamma(N+1)$ has the above fall-off property.

Note that the truncation to an effectively compact phase space does not depend on taking any semiclassical limit. It is clear that the support of $\mathcal{O}_N(z,\bar{z})$, irrespective of the original $\hat{O}$, will be confined to $r\le r_0$. The geometry of the quantum phase space is therefore that of a disc. We have proved this result here for the harmonic oscillator, given by (3.1), but similar results hold for other finite dimensional Hilbert spaces.

Another way to see the appearance of the fuzzy disc is to compute the Husimi distribution for any basis of states in $\mathcal{H}_N$. For the harmonic oscillator example (3.1), using (B.5) we see that the Husimi distribution in state $|j\rangle$ is concentrated around $r=\sqrt{j}, j=0,1,2,\dots,N-1$. The state of the maximum size, with $j=N-1$, has an approximate radius r_0. The existence of the maximum size of the Husimi distribution can be easily proved for an arbitrary linear combination of the basis states by a simple generalization of the above argument.

Coherent states in $\mathcal{H}_N$. There is another useful basis of states for $\mathcal{H}_N$, the modified coherent states (see, e.g. [32], eq. (4.16)) defined using the projection P_N (see (3.2))

$$|z, N\rangle = \frac{P_N|z\rangle}{||P_N|z\rangle||}$$

which, for the harmonic oscillator example, (3.1), becomes

$$|z, N\rangle = \frac{1}{(\mathrm{Exp}_N[|z|^2])^{1/2}} \sum_{k=0}^{N-1} \frac{z^k}{k!} \left(\alpha^\dagger\right)^k |0\rangle \tag{3.5}$$

where $\mathrm{Exp}_N[x] \equiv \sum_{k=0}^{N-1} \frac{x^k}{k!}$. The modified coherent states satisfy completeness relations

$$\int d^2 z e^{-|z|^2} |z, N\rangle\langle z, N| = P_N. \tag{3.6}$$

The Husimi distribution (regarded in the full phase space of $\mathcal{H}_\infty$) of any $|z, N\rangle$ is

$$H_{z,N}(w, \bar{w}) = \frac{e^{-|w|^2}}{\mathrm{Exp}_N[|z|^2]} |\mathrm{Exp}_N[\bar{w} z]|^2. \tag{3.7}$$

which falls off exponentially beyond $|w| = \sqrt{N}$ irrespective of the specific z (it has a peak around $w \sim z$ if $|z| \leq \sqrt{N}$).

In terms of the modified coherent states, the Husimi distribution in a state $|\psi\rangle$, defined in (B.2)), gets modified to

$$H(z, \bar{z}, N) = |\langle z, N|\psi\rangle|^2 \tag{3.8}$$

"Fuzzy" coordinate space. It is interesting to note that at finite N (and finite $\hbar$) even the coordinate space is "fuzzy". This is in the sense that, for any 'polarization' (x, p) of the phase space, localization in x ($\Delta x = 0$) requires $\Delta p = \infty$, which is impossible for a compact phase space. More precisely, the wavefunction $\delta(x - x_0)$ cannot generically be built by superposing a finite number of wavefunctions $\chi_i(x) = \langle x|i\rangle$, $i = 1, \ldots, N$. Indeed, the projection in (3.2) implies that $|x\rangle$ is replaced by the state $|x\rangle_N \equiv P_N|x\rangle$ which has the following position-space wavefunction

$$\langle y|x_N\rangle = \sum_i \langle y|i\rangle\langle i|x\rangle = \sum_{i=1}^N \chi_i^*(y)\chi_i(x) \tag{3.9}$$

This approaches $\delta(x - y)$ only in the limit $N \to \infty$.

3.2 Second quantization: the bosonic phase space density

Let us define the following second quantized field

$$\phi(x) \equiv \sum_{i=1}^N a_i \chi_i(x), \quad \phi^\dagger(x) \equiv \sum_{i=1}^N a_i^\dagger \chi_i^*(x) \tag{3.10}$$

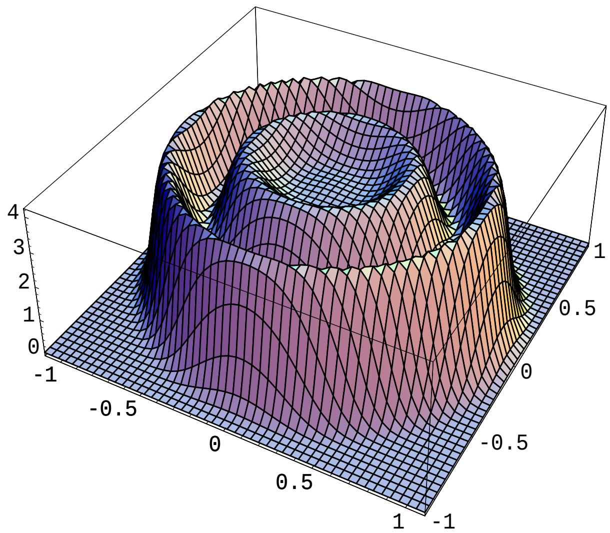

Figure 1: The bosonic phase space density $\langle W_B(x,p)\rangle$, in the state (2.5), as a function of the (x,p) plane. In the LLM example, this corresponds to the density of giant gravitons in the (x_1, x_2) plane.

The bosonic Fock space state (2.5) then has the wavefunction

$$\begin{aligned}&\langle x_1, \dots, x_M | r_1, \dots, r_N\rangle \\ &= \sum_{\sigma \in P_M} \frac{1}{\sqrt{M!}} \left[\prod_{i=1}^{r_1} \chi_{\sigma(1)}(x_i) \prod_{i=r_1+1}^{r_2} \chi_{\sigma(2)}(x_i) \cdots \prod_{i=r_1+r_2+\cdots+r_{N-1}+1}^{M} \chi_{\sigma(N)}(x_i)\right]\end{aligned} \tag{3.11}$$

where $|x_1, \dots, x_M\rangle \equiv \prod_{l=1}^{M} \phi^\dagger(x_l)\ |0\rangle$, $M = r_1 + \cdots + r_N$. Just like $|x\rangle$ is outside the single-particle Hilbert space $\mathcal{H}_N$, the basis $|x_1, \dots, x_M\rangle$ is outside of the Fock space built out of states (2.5). The more appropriate wavefunctions are products of $|x\rangle_N$ as in (3.9). Nevertheless, (3.10) is a useful definition to have.

The second quantized (Wigner) phase space density is given in terms of (3.10) as (cf. (B.10))

$$\hat{W}_B(x,p) = \int d\eta\ e^{i\eta p}\ \phi^\dagger(x+\eta/2)\ \phi(x-\eta/2) \tag{3.12}$$

The expectation value of $\hat{W}_B(x,p)$ in the state $|r_1, \dots, r_N\rangle$ is easily computed:

$$\langle \hat{W}_B(x,p)\rangle = \sum_{i=1}^{N} r_i W_i(x,p), \tag{3.13}$$

where

$$W_i(x,p) = \int d\eta\ e^{i\eta p}\ \chi_i(x-\eta/2)\ \chi_i^*(x+\eta/2) \tag{3.14}$$

represents the Wigner density for the individual state $|i\rangle$. The semiclassical picture of (3.13) for a typical state is described in figure 1.

For the harmonic oscillator example, one can use (B.5) to evaluate (3.13). The plot of (3.13) in the (x,p) plane looks like "a rugged circular cake" with a maximum diameter

r_0 given by (3.4) and with circular ridges of heights r_i at radii $\sim \sqrt{i}$. For states which are arbitrary linear combinations of (2.5) the phase space density still has the shape of a rugged cake of the same maximum diameter r_0, but with the circular ridges generically replaced by dips and bumps not necessarily maintaining the circular symmetry.

In the context of LLM, these plots depict the density of giant gravitons in the (x_1, x_2) plane (see section 4.1).

Another way of seeing the "rugged cake" is to compute

$$\langle z_1^1, \ldots, z_{r_1}^1, z_1^2, \ldots, z_{r_2}^2, z_1^N, \ldots, z_{r_N}^N | r_1, \ldots, r_N \rangle \sim \mathrm{Sym}\left[\prod_{i=1}^{N}\prod_{j=1}^{r_i}(z_j^i)^i \exp - \left[\sum_{i=1}^{N}\sum_{j=1}^{r_i}|z_j^i|^2/2\right]\right] \tag{3.15}$$

which shows that there are r_k particles at radius $\sqrt{k}$ for each $k = 1, \ldots, N$. The notation "Sym" represents symmetrization over all the z's.

A more appropriate second quantized phase space density is (cf. (B.10))

$$\begin{aligned} H_{B,N}(z,\bar{z}) &= \phi(z,\bar{z},N)\phi^\dagger(z,\bar{z},N) \\ \phi(z,\bar{z},N) &\equiv \sum_{i=1}^{N} a_i \langle z, N | i \rangle \end{aligned} \tag{3.16}$$

which are finite N versions of (B.10).

3.3 Lattice interpretation

Another, perhaps more appropriate, description of the coordinate space is in terms of a finite lattice. An easy example is when $\mathcal{H}_N$ is built out of the first N levels of a harmonic oscillator (see eq. (3.1)). Consider the radial polarization of the phase space where the angle φ is the coordinate and the hamiltonian $r^2/2 = -i\partial/\partial\varphi$ is the conjugate momentum. The ultra-violet cut-off in energy or the radius (see (3.4)) implies that there is a finite lattice cutoff in φ (this, together with the infra-red cut-off coming from the compactness of φ, gives a finite lattice, as appropriate for a finite dimensional Hilbert space $\mathcal{H}_N$). Thus φ takes discrete values $\varphi_\mu = \mu\epsilon,\ \epsilon = 2\pi/N,\ \mu = 0, 1, \ldots, N-1,\ \varphi_\mu \equiv \varphi_{\mu+N} = \varphi_\mu + 2\pi$. The associated 'position eigenstates' $|\varphi_\mu\rangle$ are defined as discrete Fourier transforms of the basis states in (3.1):

$$|\varphi_\mu\rangle \equiv \sum_{k=1}^{N} e^{-i\frac{2\pi}{N}\mu k}|k-1\rangle \tag{3.17}$$

The lattice description of the coordinate space is more appropriate than the continuum description since with a continuous variable φ the associated states $|\varphi\rangle$ are typically outside of $\mathcal{H}_N$. In contrast, $|\varphi_\mu\rangle$ are all in $\mathcal{H}_N$, by definition.

The second quantized field operator $\phi^\dagger(\varphi_\mu) \equiv \phi_\mu^\dagger$, which creates a particle on the μth site, is defined by

$$\phi_\mu^\dagger \equiv \frac{1}{\sqrt{N}} \sum_{k=1}^{N} e^{-i\frac{2\pi}{N}\mu k} a_k^\dagger \tag{3.18}$$

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Its conjugate, ϕ_μ, destroys a particle on the μth site. It is easy to show that these field operators also satisfy the harmonic oscillator algebra

$$
\begin{aligned}
[\phi_\mu, \phi_\nu^\dagger] &= 1 \qquad \text{if} \qquad \mu = \nu \qquad \text{mod} \qquad N \\
&= 0 \qquad \text{otherwise.}
\end{aligned} \tag{3.19}
$$

In terms of these field operators, the N-level lattice hamiltonian may be obtained from (2.23) by using the following relation:

$$
\sum_{i=k}^{N} a_i^\dagger a_i = \frac{N-k+1}{N} \sum_{\mu=0}^{N-1} \phi_\mu^\dagger \, \phi_\mu + \frac{1}{N} \sum_{\mu \neq \nu = 0}^{N-1} \left[\frac{e^{i\frac{2\pi}{N}(\mu-\nu)k} - e^{i\frac{2\pi}{N}(\mu-\nu)}}{1 - e^{i\frac{2\pi}{N}(\mu-\nu)}} \right] \phi_\mu^\dagger \, \phi_\nu \tag{3.20}
$$

The hamiltonian is essentially the charge Q_1, see equation (3.22) below. Notice that the second term in the above equation gives rise to long-range interactions on the lattice.

Our lattice models inherit higher conserved charges from the underlying free fermion system. There are exactly N conserved charges, which arise due to the conservation of the individual energies of the non-interacting fermions. It is more usual to write these conserved charges as

$$
Q_n = \sum_{m=0}^{\infty} \{\mathcal{E}(m)\}^n \; \psi_m^\dagger \, \psi_m. \tag{3.21}
$$

Here n is any positive integer, but the number of independent charges is only N since the values of these charges in any N-fermion state can be expressed in terms of only the N independent energies of the occupied levels. In the lattice formulation of the bosonized theory, these charges translate into the operators

$$
Q_n = \sum_{k=1}^{N} \left\{ \mathcal{E}\left(N - k + \frac{N-k+1}{N} \sum_{\mu=0}^{N-1} \phi_\mu^\dagger \, \phi_\mu + \frac{1}{N} \sum_{\mu \neq \nu = 0}^{N-1} \left[\frac{e^{i\frac{2\pi}{N}(\mu-\nu)k} - e^{i\frac{2\pi}{N}(\mu-\nu)}}{1 - e^{i\frac{2\pi}{N}(\mu-\nu)}} \right] \phi_\mu^\dagger \, \phi_\nu \right) \right\}^n \tag{3.22}
$$

By construction these higher charges exist for any potential in which the fermions are moving. In particular, they also exist for fermions moving in a harmonic oscillator potential, which is relevant for the half-BPS sector of $\mathcal{N} = 4$ super Yang-Mills theory. This raises the possibility of some connection of our lattice models with the integrable spin-chain models of $\mathcal{N} = 4$ super Yang-Mills theory [4] which have been a recent focus of study in connection with AdS/CFT duality. What this connection might be is not clear to us, but investigating this possibility could be worthwhile.

We end this section by mentioning that an explicit expression for the lattice hamiltonian can be given for an equally spaced spectrum of the form $\mathcal{E}(m) = c_1 \; m + c_2$. This is relevant for the half-BPS sector mentioned above. The hamiltonian turns out to be

$$
H_{\text{equal-spacing}} = \frac{c_1}{2}(N+1) \sum_{\mu=0}^{N-1} \phi_\mu^\dagger \, \phi_\mu + \frac{c_1}{2} \sum_{\mu \neq \nu = 0}^{N-1} \left[1 - i \; \cot \frac{\pi}{N}(\mu - \nu) \right] \phi_\mu^\dagger \, \phi_\nu + H_{\text{vac}} \tag{3.23}
$$

[4] See, for example, the paper by Beisert and Staudacher [33] for a summary of the current status of the subject and for references to recent literature.

where $H_{\rm vac} = \frac{c_1}{2} N(N-1) + c_2 N$ is the ground state energy. This hamiltonian has long-range interactions. Moreover, unlike in the case of standard SYM spin-chains [33], here there can be multiple excitations at any site since the variables at each site are harmonic oscillators. Whether the reformulation of the half-BPS sector as the spectrum of this lattice hamiltonian has any new insights to offer remains to be seen.

3.4 More on second quantization

In view of the lattice interpretation of coordinate space, it may be appropriate to introduce discrete Wigner/Husimi phase space distributions [34] $W_B(i,j), H_B(i,j)$ in both the first and second quantized formalism (the precursors of this concept are (3.8), (3.16)). We will not go into the full details here, but consider only the diagonal elements which turn out to be

$$\hat{W}_B(i,i) = \hat{\Phi}_B(i,i) = a_i^\dagger a_i \tag{3.24}$$

Using (2.21), we can relate these to $\psi_n^\dagger \psi_n$ which, when evaluated on wavefunctions of the form (2.2), get related to the fermionic Wigner density $\hat{W}_F(x,p)$ (see (B.8), also footnote 5),

$$\langle \hat{W}_F(x,p) \rangle = \sum_n \langle \psi_n^\dagger \psi_n \rangle W_n(x,p) \tag{3.25}$$

Here W_n are the single-particle Wigner distributions. For the corresponding states (2.5) we find (see (3.13))

$$\langle \hat{W}_B(x,p) \rangle = \sum_i \langle a_i^\dagger a_i \rangle W_i(x,p) \tag{3.26}$$

Combining all this with (2.21) we get for these special states (monomials of the form (2.2))

$$\langle \hat{W}_F(x,p) \rangle = \sum_n \sum_{k=1}^N \left\langle \delta \left(\sum_{i=k}^N \hat{W}_B(i,i) + N - n - k \right) \right\rangle W_n(x,p) \tag{3.27}$$

In case of (3.1), $\langle \hat{W}_B \rangle, \langle \hat{W}_F \rangle$ for these states are functions of only $x^2 + p^2$ ("circular configurations") in which case we get a relation between the bosonic and fermionic phase space densities

$$\langle \hat{W}_F(r) \rangle = \sum_n \sum_{k=1}^N \delta \left(\sum_{i=k}^N \left\{ \int [dr'] \langle \hat{W}_B(r') \rangle W_i(r') \right\} + N - n - k \right) W_n(r) \tag{3.28}$$

where we have taken the semiclassical limit to perform the average inside the δ-function.

In the next section, we will interpret this as a relation between the LLM metric and the giant graviton density.

3.5 Action for W_B

For noninteracting fermions, the second quantized action for the fermions is given by the Kirillov coadjoint orbit action written in terms of the Wigner phase space density $\hat{W}_F$ [20, 21, 35, 5, 6] [5]

$$S[W_F] = S_{\rm sympl}[W_F] + S_{\rm ham}[W_F],$$

[5] In these references the notation $u(q,p)$ was used in place of $W_F(q,p)$; notations in this paper are explained in appendix B.

$$S_{\rm sympl} = \int dt\ ds \int \frac{dqdp}{2\pi\hbar}\ W_F(q,p,t,s) *_w \{\partial_t W_F, \partial_s W_F\}_{MB},$$
$$S_{\rm ham} = \int dt\ H_F = \int dt \int \frac{dqdp}{2\pi\hbar}\ W_F(q,p,t) *_w h(q,p) \tag{3.29}$$

Here the action is written in the first order form, similar to the single-particle action $\int dt\ (p\dot{q} - H(p,q))$, with $W_F(q,p)$ itself playing the role of the phase space coordinates (p,q). The notation "sympl" denotes the symplectic form $(p\dot{q})$ term and "ham" denotes the hamiltonian term. The variable s denotes an extension of $W_F(q,p,t)$ to $W_F(q,p,t,s)$, devised to enable us to write down the symplectic term. The physical trajectory is parametrized by t at some $s = s_0$ and the equation of motion is independent of the s-extension because the symplectic form is closed. See [35, 5, 6] for details.

Using the bosonization formulae in section 2 and appendix A one gets an action for the bosons in terms of $W_B(q,p)$. For equally spaced fermionic levels we can use (3.28) to get an action of the form

$$S[W_B] = S_{\rm sympl}[W_B] + \int dt \int \frac{dqdp}{2\pi\hbar}\ W_B(q,p,t) *_w h_B(q,p) \tag{3.30}$$

where h_B is an equally spaced finite-level (N) hamiltonian. The symplectic form can be explicitly written down, though we will not do so here.

4. Application of our bosonization to LLM

Review. As discussed in the Introduction, [2, 3] made the observation that a half-BPS sector of $\mathcal{N} = 4$ super Yang Mills theory is described effectively by a theory of free fermions moving in a simple harmonic oscillator potential. The semiclassical fermion phase space is described by droplets of uniform density in two dimensions. Ref. [1] uncovered such a droplet structure also in the corresponding sector of type IIB supergravity in asymptotically $AdS_5 \times S^5$ spacetimes. It was noted in [2, 3, 12] that states of the fermion theory should have a bosonic description in terms of giant gravitons, since the latter are known to correspond to operators of the Yang Mills theory which can be written in terms of the fermions [4]. The identification between the fermionic and the bosonic states was explicitly stated in [12] as a one-to-one map

$$|f_1, \ldots, f_N\rangle \leftrightarrow |r_1, \ldots, r_N\rangle \tag{4.1}$$

where

$$r_N = f_1,$$
$$r_k = f_{N-k+1} - f_{N-k} - 1, \qquad k = 1, 2, \ldots, N-1 \tag{4.2}$$

This maps the fermionic hamiltonian

$$H_F = \sum_{n=0}^{\infty} (n + 1/2) \psi_n^\dagger \psi_n, \tag{4.3}$$

with excitation spectrum given by $E = \sum_{k=1}^{N}(f_k - k + 1)$, to a bosonic hamiltonian

$$H_F - N^2/2 = H_B = \sum_{i=1}^{N} i a_i^\dagger a_i, \tag{4.4}$$

with excitation spectrum given by $E = \sum_{k=1}^{N} k r_k$. A few examples of the map (4.2) are

$$\begin{aligned}
|F_0\rangle \equiv |0, 1, \ldots, N-1\rangle &\;\mapsto |0\rangle_B \\
\psi_N^\dagger \psi_0 |F_0\rangle \equiv |1, 2, \ldots, N\rangle &\;\mapsto a_N^\dagger |0\rangle_B \\
\psi_N^\dagger \psi_{N+1}^\dagger \psi_1 \psi_0 |F_0\rangle \equiv |2, 3, \ldots, N+1\rangle &\;\mapsto \left((a_N^\dagger)^2/\sqrt{2}\right)|0\rangle_B \\
\psi_{N+1}^\dagger \psi_N^\dagger \psi_2 \psi_0 |F_0\rangle \equiv |1, 3, \ldots, N, N+1\rangle &\;\mapsto a_N^\dagger a_{N-1}^\dagger |0\rangle_B
\end{aligned} \tag{4.5}$$

The second equation assigns a single hole at the lowest level (costing energy N) to a single bosonic particle at level N. The 4th equation assigns two holes at levels 0 and 2 (total energy cost $2N-1$) to two bosonic particles, at levels N and $N-1$.

The rationale behind the map (4.2) is as follows. A fermion configuration can be specified in terms of holes created in the Fermi sea. The idea is to regard a hole (together with the upward shift of the Fermi sea to make space for it) as a bosonic excitation. The correspondence between giant gravitons and gauge invariant operators suggests an identification of these bosonic excitations with giant gravitons.

Our operator map and Exactness of the Bose-Fermi equivalence. The above rationale of treating a hole as a boson is somewhat intuitive, so let's see how it holds in specific examples. Consider the second equation of (4.5). Here the hole corresponds to the excitation $A_N^\dagger \equiv \Phi_{N0} = \psi_N^\dagger \psi_0$, which satisfies the algebra (see (B.6))

$$[A_N, A_N^\dagger] = \psi_0^\dagger \psi_0 - \psi_N^\dagger \psi_N \tag{4.6}$$

This evaluates to 1 on $|F_0\rangle$ but not in general. Indeed these operators are related to the fermion phase space density which satisfy the W_∞ algebra but not satisfy the Heisenberg algebra.

This might seem to suggest that (4.2) may not hold operatorially [36], that is, the "hole" operators may not satisfy the usual bosonic commutation rules, after all. On the other hand, it is also known [12] that with this map the fermionic and bosonic partition functions agree. The only way to settle which possibility is realized is to try to see if one can deduce operator maps from (4.2). The operator maps described in this paper (section 2, appendix A and section 5) indeed precisely fit the bill [6]. The precise bosonic excitation is more complicated than the naive guess of (4.6) and is such that it does satisfy the Heisenberg algebra. Indeed the Heisenberg algebra is implied by the the fermion anticommutation relations (and vice versa) and the bose-fermi equivalence is exact.

[6] We first deduced the operator map for the LLM system, but as mentioned in section 2, they hold for fermions moving in an arbitrary potential and even in the presence of fermion-fermion interactions

4.1 Giant graviton phase space density

The exact bosonic operators $a_i, a_i^\dagger$ are clearly related to creation or destruction of giant gravitons [2, 3, 1, 12, 5]. To see this in detail, we now come back to the bosonic phase space discussed in the earlier section.

The correspondences with the last section are:

1. The states $|i\rangle \in \mathcal{H}_N$ correspond to giant gravitons in energy eigenstates.

2. The modified coherent states $|z, N\rangle, |z| < N$ describe a giant graviton state localized near the point z.

3. The giant graviton energy levels are equally spaced, as in the harmonic oscillator example (3.1). Thus, the phase space density of giant gravitons (i.e. the density of giant gravitons in the (x_1, x_2) plane)[7] has (see eq. (3.13) and below) the geometry of a rugged cake[8], as discussed in section 3.2 (see figure 1). Such a geometry (with heights r_i at radii $\sim \sqrt{i}$) accords with the picture of r_i giant gravitons moving in the i-th orbit. This adds to the evidence that the bosons $a_i, a_i^\dagger$ indeed represent giant gravitons.

4. Arbitrary LLM 'droplet' geometries correspond, in a one-to-one fashion, to the "rugged cake" geometries of giant gravitons. It is interesting to note that the giant gravitons *never leave the original circular region (3.4) representing* $AdS_5 \times S^5$, even for arbitrary LLM geometries.

5. For circular configurations, the giant graviton phase space density in the semiclassical limit is directly related to the fermion phase density via (3.28). Because of the relation between the fermion phase space density and the LLM metric, (3.28) also expresses the LLM metric in terms of the giant graviton phase space density in the semiclassical limit. Section 4.2 will discuss the finite N version of this relation where we will write down exact expressions for gravitons in terms of the oscillators $a_i, a_j^\dagger$.

Remarks: a. The bosonization formula in section 2 automatically incorporate the fact that although multiple fermions cannot occupy the same energy levels, the giant gravitons can. In the simple examples like (4.5), where the r.h.s. of the third line describes two giant gravitons in the N-th orbit, the equivalent fermionic description on the l.h.s. describes a spreading out. In general such effects are encoded in the operator maps.

b. In [5], only non-overlapping giant gravitons were considered, and agreement found with the W-action (3.29). When overlapping giant gravitons are considered, the data consist of not only the centres of mass of the giant gravitons, but the "heights" (how many on top of each other). It would be an interesting exercise to obtain (3.30) from considering such overlapping giant gravitons.

[7] The identification of the (x_1, x_2) plane of LLM with the phase space of half-BPS giant gravitons was made in [5].

[8] There is a subtlety, however, about the origin of the "cake". The giant gravitons at the "North pole" have zero energy. If we are not interested in keeping track of the total number of giant gravitons, we can choose to ignore all such giant gravitons and therefore ignore the harmonic oscillator ground state. Our P_N (see (3.2)) will therefore consist of the states $|i\rangle,\ i = 1, 2, \ldots, N$. The formulae in the last section will have to be correspondingly modified, and in this convention, the rugged cake will have a "dip" in the middle.

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4.2 LLM gravitons and our bosons

Consider the single trace operators of the boundary theory. In the fermionic realization, these operators correspond to [4]

$$\beta_m^\dagger = \sum_{n=0}^{\infty} \sqrt{\frac{(m+n)!}{2^m n!}} \;\; \psi_{n+m}^\dagger \; \psi_n, \qquad\qquad m = 1,\, 2,\, \ldots, \infty \tag{4.7}$$

It is easy to check that $[\beta_m^\dagger, \beta_n^\dagger] = 0 = [\beta_m, \beta_n]$, but that $[\beta_m, \beta_n^\dagger] \neq \delta_{mn}$. In fact, in terms of our bosonic oscillators which do satisfy the standard oscillator algebra, (2.4), the single trace operators have complicated expressions, involving creation as well as annihilation operators. The operator expression, which can be obtained by using the bosonization formula, (A.8), is quite messy, but the corresponding "single-particle" state, which is obtained by acting on the fermi vacuum, has a simple enough expression:

$$\begin{aligned}\beta_m^\dagger \; |F_0\rangle = \; & \sum_{n=2}^{N} (-1)^{n-1} \sqrt{\frac{(N+m-n)!}{2^m (m-n)!(N-n)!}} \; \theta_+(m-n) \; {a_1^\dagger}^{m-n} \; a_n^\dagger \; |0\rangle_B \\ & + \sqrt{\frac{(N+m-1)!}{2^m m!(N-1)!}} \; {a_1^\dagger}^{m} \; |0\rangle_B .\end{aligned} \tag{4.8}$$

We see that even a "single-particle" state has in general many excitations of our elementary bosons. For $m < N$, because of the theta-function the sum in the first term on the right-hand side of (4.8) terminates at $n = m$. So in this case the highest energy creation operator that appears on the right-hand side is $a_m^\dagger$ and it appears by itself. If $m \ll N$, the leading term on the right-hand side in $1/N$ expansion has an overall factor of $N^{m/2}$, which is the correct large-N normalization for a single trace operator. We thus begin to see how the usual picture of collective excitations arises in the large-N limit for low-energy states. On the other hand, for any $m > N$ the highest energy creation operator that appears in "single-particle" states is $a_N^\dagger$ and it appears together with other excitations. This is a reflection of the fact that a single trace operator in the boundary theory of higher than Nth power of a matrix is not independent since it can be rewritten in terms of products of lower traces. Let us now explain the above comments in greater detail.

4.2.1 Fuzzy gravitons

From (4.7) we get

$$\begin{aligned}\beta_1^\dagger|F_0\rangle \; &= \sqrt{\frac{N}{2}} a_1^\dagger |0\rangle_B \\ \beta_2^\dagger|F_0\rangle \; &= -\frac{1}{2}\sqrt{N(N-1)} a_2^\dagger |0\rangle_B + \frac{1}{2}\sqrt{\frac{N(N+1)}{2}} (a_1^\dagger)^2 |0\rangle_B \\ &\;\;.. \\ \beta_{N+1}^\dagger \; |F_0\rangle = \; & \sum_{n=2}^{N} (-1)^{n-1} \sqrt{\frac{(N+m-n)!}{2^m (m-n)!(N-n)!}} \;\; {a_1^\dagger}^{N+1-n} \; a_n^\dagger \; |0\rangle_B \\ & + \sqrt{\frac{(2N)!}{2^{N+1}(N+1)!(N-1)!}} \; {a_1^\dagger}^{N+1} \; |0\rangle_B .\end{aligned} \tag{4.9}$$

Consider the first equality. Taking inner product with $|z\rangle$ on both sides, we see that

$$\langle z|\beta_1^\dagger|F_0\rangle = \sqrt{\frac{N}{2}}\langle z|a_1^\dagger|0\rangle_B \sim z\exp[-|z|^2/2] \tag{4.10}$$

This is the wave-function (in the coherent state basis) of the "graviton" of unit energy. The "graviton" of energy $m \ll N$, always has a single particle component $\sim z^m \exp[-|z|^2/2]$. These correspond to the gravitons wave-functions of [37]. For $m > 1$, however, they also have multi-particle components, e.g. for $m = 2$, there is a two particle component with wavefunction $\langle z_1, z_2|(a_1^\dagger)^2|0\rangle_B \sim z_1 z_2 \exp[-(|z_1|^2 + |z_2|^2)/2]$. For $m > N$ there are no single-particle components of the wave-functions.

Remarks: 1. It is easy to see from (4.9) that "gravitons" of energy $> N$ do not exist as single particles and are composed of multiple $a_i^\dagger$ modes of lower energy. This suggests that the existence of an infinite number of gravitons, a hallmark of commutative gravity, is only an approximation valid in the large N limit. For finite N, the oscillators $a_i, a_i^\dagger$, which are by definition N in number and hence independent of each other, provide a more appropriate basis to describe the geometry.

2. Finite number of independent metric fluctuations is an indication of noncommutative geometry. Thus, e.g., metric fluctuations on a fuzzy sphere will involve spherical harmonics only up to rank proportional to the radius of the fuzzy sphere.

3. Formulae such as (3.4) suggest that the large N limit involves nonperturbative effects of the form $\exp[-1/\hbar] \sim \exp[-N]$.

4. At finite N, superscript effective action should receive interesting corrections from the "compactness" of the phase space which is captured by the difference between the Moyal star product and the ordinary product appearing in the W_F action (3.29) [5, 6] and the corresponding structure in the W_B action (3.30).

5. The second bosonization

Consider a second system of bosons each of which can occupy a state in an infinite Hilbert space $\mathcal{H}_B$. Suppose we choose a basis $|m\rangle, m = 0, \ldots, \infty$ of $\mathcal{H}_B$. In the second quantized notation we introduce creation (annihilation) operators $b_m^\dagger$ (b_m) which creates (destroys) a particle in the state $|m\rangle$. These satisfy the commutation relations

$$[b_m, b_n^\dagger] = \delta_{mn}, \quad m, n = 0, \ldots, \infty \tag{5.1}$$

A state of this bosonic system is given by (a linear combination of)

$$\prod_{k=0}^{\infty} \frac{(b_k^\dagger)^{n_k}}{\sqrt{n_k!}}|vac\rangle\,. \tag{5.2}$$

Now consider the subspace of the Hilbert space spanned by states with the restriction $\sum_{k=0}^{\infty} n_k = N$. We label a state of this type by $|s_1, s_2, \ldots, s_N\rangle$ where s_i are non-increasing set of integers representing the energies of N bosons. In [12] a map between these states and the fermionic states was proposed

$$s_{N-i} \mapsto f_{i+1} - i, \quad i = 0, 1, \ldots, N-1 \tag{5.3}$$

The rationale behind this map was to regard the "particle" excitations (which correspond to dual giant gravitons) as bosons. In the following subsection we describe the operator equivalence of this system with the N-fermion system.

5.1 The map between fermions and 'dual-giant' bosons

Let us start by rewriting the Hilbert space of the fermion system into a disjoint union of subspaces spanned by states with a fixed number of particle excitations.

$$\mathcal{H}^{(F)} = \cup_{k=0}^{N} \mathcal{H}_k^{(F)} \tag{5.4}$$

where $\mathcal{H}_{N-k}^{(F)} = \mathrm{Span}\{|0, 1, \ldots, k-1, f_{k+1}, \ldots, f_N\rangle : f_{k+1} \neq k\}$. The operator that distinguishes the states belonging to different subspaces $\mathcal{H}_k^{(F)}$ is $\hat{\nu} := \sum_{k=1}^{\infty} b_k^\dagger b_k$ which translates to

$$\hat{\nu} = \sum_{k=1}^{\infty} b_k^\dagger b_k = N - \sum_{k=1}^{N} \prod_{m=0}^{k-1} \psi_m^\dagger \psi_m \,. \tag{5.5}$$

Further, define

$$\begin{aligned}
\hat{s}_p &= \sum_{k=1}^{\infty} (k-1) \sum_{l=0}^{p-1} \delta\left(\sum_{i=k}^{\infty} b_i^\dagger b_i - l\right) \sum_{m=p}^{N} \delta\left(\sum_{j=k-1}^{\infty} b_j^\dagger b_j - m\right), \\
\hat{f}_p &= \sum_{k=0}^{\infty} k\, \delta\left(\sum_{i=0}^{k-1} \psi_i^\dagger \psi_i - p + 1\right) \delta\left(\sum_{j=0}^{k} \psi_j^\dagger \psi_j - p\right).
\end{aligned} \tag{5.6}$$

and

$$\begin{aligned}
\psi_{f_p} &= \sum_{k=0}^{\infty} \psi_k \delta\left(\sum_{i=0}^{k-1} \psi_i^\dagger \psi_i - p + 1\right) \delta\left(\sum_{j=0}^{k} \psi_j^\dagger \psi_j - p\right) \\
b_{s_p} &= \sum_{k=1}^{\infty} b_{k-1} \sum_{l=0}^{p-1} \delta\left(\sum_{i=k}^{\infty} b_i^\dagger b_i - l\right) \sum_{m=p}^{N} \delta\left(\sum_{j=k-1}^{\infty} b_j^\dagger b_j - m\right)
\end{aligned} \tag{5.7}$$

for $p = 1, 2, \ldots, N$. The corresponding $\psi_p^\dagger$ and $b_{s_p}^\dagger$'s can be obtained by taking the hermitian conjugates. Since we are to keep the total number of particles fixed we only consider the operators of the type $b_k^\dagger b_m$. Using eq. (5.1) we have

$$[b_k^\dagger b_m,\ b_p^\dagger b_q] = \delta_{mp} b_k^\dagger b_q - \delta_{kq} b_p^\dagger b_m \tag{5.8}$$

which is again a W_∞-algebra. We want to find the operator correspondences between operators of the kind $b_k^\dagger b_m$ on the bosonic side and $\psi_k^\dagger \psi_m$ type operators on the fermionic side. We present below the expressions of these operators for the special cases of $N = 1$, 2. The generalization to arbitrary N is a lengthy but straightforward exercise.

For $N = 1$, we can associate $b^\dagger_k b_m$ with $\psi^\dagger_k \psi_m$ (and $b_k b^\dagger_m$ with $2\delta_{km} - \psi_k \psi^\dagger_m$) and the algebra (5.8) follows immediately.

For $N = 2$ case we first seek operators $b^\dagger_k b_0$ ($b^\dagger_0 b_k$) which create (annihilate) a particle excitation that takes a state in $\mathcal{H}^{(F)}_m$ to one in $\mathcal{H}^{(F)}_{m+1}$ ($\mathcal{H}^{(F)}_{m+1}$ to one in $\mathcal{H}^{(F)}_m$). To find these let us observe that each state of the subspace $\mathcal{H}^{(F)}_k$ is a linear combination of states with k excited bosons (and $N - k$ in the ground state) $|s_1, s_2, \ldots, s_k, s_{k+1} = 0, \ldots, s_N = 0\rangle$ with $k \leq N$. The operator $b^\dagger_k b_0$ excites a particle from level-0 to level-k. This operator takes $|s_1 = 0, \ldots, s_N = 0\rangle$ to $|s_1 = k, s_2 = 0, \ldots, s_N = 0\rangle$. Similarly it takes $|s_1, 0, \ldots, 0\rangle$ to $|s_1, k, 0, \ldots 0\rangle$ if $k \leq s_1$ or $|k, s_1, 0, \ldots, 0\rangle$ if $s_1 \leq k$ and so on.

We find the following expressions for $b^\dagger_k b_0$ and $b^\dagger_0 b_k$:

$$
\begin{aligned}
b^\dagger_k b_0 &= \sqrt{2}\,\psi^\dagger_{k+1}\psi_1\,\delta_{\hat\nu} + \Bigg[\sum_{l=1}^{k-1}\psi^\dagger_{k+1}\psi_0\psi^\dagger_{k+l}\,\psi_{k-l+1}\,\delta_{k-\hat f_2-l+1} \\
&\qquad\qquad + \psi^\dagger_k\psi_0\left(\sum_{l=1}^{\infty}\delta_{\hat f_2-k-l-1} + \sqrt{2}\,\delta_{k-\hat f_2+1}\right)\Bigg]\,\delta_{\hat\nu-1}, \\
b^\dagger_0 b_m &= \psi^\dagger_1\,\psi_{m+1}\,\delta_{m-\hat f_2+1}\,\delta_{\hat\nu-1} + \Bigg[\psi^\dagger_0\,\psi_{\hat f_2}\,\psi_{\hat f_1}\,\psi^\dagger_{\hat f_1+1}\,\delta_{\hat f_2-m-1}\,(1-\delta_{\hat f_2-\hat f_1-1}) \\
&\qquad\qquad + \psi^\dagger_0\,\psi_m\,\delta_{\hat f_1-m}\,(1+\sqrt{2}\,\delta_{\hat f_2-\hat f_1-1})\Bigg]\,\delta_{\hat\nu-2}
\end{aligned}
\tag{5.9}
$$

for $k, m \geq 1$. The general operators of the type $b^\dagger_k b_m$ can be generated out of the above ones using

$$
b^\dagger_k b_m = \delta_{km}\, b^\dagger_0 b_0 + [b^\dagger_k b_0, b^\dagger_0 b_m]. \tag{5.10}
$$

The fermion bilinear operators turn out to be:

$$
\psi^\dagger_k\psi_k = (\delta_k + \delta_{k-1})\,\delta_{\hat\nu} + (\delta_k + \delta_{k-\hat s_1-1})\,\delta_{\hat\nu-1} + (\delta_{k-\hat s_2} + \delta_{k-\hat s_1-1})\,\delta_{\hat\nu-2} \tag{5.11}
$$

and

$$
\begin{aligned}
\psi^\dagger_{k+1}\psi_k &= \frac{1}{\sqrt{2}}b^\dagger_1 b_0\,\delta_{k-1}\,\delta_{\hat\nu} + (b^\dagger_1 b_0\,(1-\delta_{\hat s_1})\delta_k + b^\dagger_k\, b_{k-1}\,\delta_{k-\hat s_1-1})\,\delta_{\hat\nu-1} \\
&\quad + \left(b^\dagger_{k+1}\,b_k\,\delta_{k-\hat s_2}\,(1-\delta_{k-\hat s_1}) + b^\dagger_k\,b_{k-1}\,\delta_{k-\hat s_1-1}\left(1 + \frac{1-\sqrt{2}}{\sqrt{2}}\delta_{k-\hat s_2-1}\right)\right)\,\delta_{\hat\nu-2}, \\
\psi^\dagger_{k+n}\psi_k &= \frac{1}{\sqrt{2}}\left[b^\dagger_n b_0\,\delta_{k-1} + b^\dagger_{n-1}b^\dagger_1 b_0^2\left(1 + \frac{1-\sqrt{2}}{\sqrt{2}}\delta_{n-2}\right)\,\delta_k\right]\,\delta_{\hat\nu} \\
&\quad + \Bigg[b^\dagger_{k+n-1}\,b_{k-1}\,\delta_{k-\hat s_1-1} + \Bigg(\sum_{l=2}^{\infty} b^\dagger_n b_0\,\delta_{\hat s_1-n-l+1} \\
&\quad + \sum_{l=2}^{n-1} b^\dagger_{n-1}\,b_0\,b^\dagger_{n-l}\,b_{n-l-1}\,\delta_{n-\hat s_1-l-1}
\end{aligned}
$$

$$+\frac{1}{\sqrt{2}}((b^\dagger_{n-1})^2 b_0\, b_{n-2}\, \delta_{n-\hat{s}_1-2} + \delta_{\hat{s}_1-n} b^\dagger_n\, b_0)\, \delta_k \Bigg]\, \delta_{\hat{\nu}-1}$$
$$+\Bigg[b^\dagger_{k+n-1}\, b_{k-1}\, \delta_{k-\hat{s}_1-1} + \Bigg(\sum_{l=2}^{\infty} b^\dagger_{k+n}\, b_k\, \delta_{\hat{s}_1-k-n-l+1}$$
$$+\sum_{l=2}^{n-1} b^\dagger_{k+n-1}\, b^\dagger_{k+n-l}\, b_{k+n-l-1}\, b_k\, \delta_{k+n-\hat{s}_1-l-1}$$
$$+\frac{1}{\sqrt{2}}(b^\dagger_{k+n}\, b_k\, \delta_{\hat{s}_1-k-n} + (b^\dagger_{k+n-1})^2\, b_{k+n-2}\, b_k\, \delta_{k+n-\hat{s}_1-2})\, \delta_{k-\hat{s}_2} \Bigg]\, \delta_{\hat{\nu}-2} \tag{5.12}$$

for $n \geq 2$. The $\delta_{\hat{O}}$'s in these expressions are the same as the operator delta functions $\delta(\hat{O})$ used for the first boson in section 2. Similar expressions hold for $\psi^\dagger_k \psi_{k+n}$.

The fact that the N-fermion system is equivalent to two different bosonic systems implies that the two bosonic systems are also equivalent to each other [12] (see [38] also). As in case of the first boson, one can represent various states in this bosonic system by specifying the corresponding phase space densities. One expects that the phase space here is the same as that of a harmonic oscillator (i.e, $\mathbf{R}^2$) with the total number of particles being equal to N. Since they are bosons, each Planck cell can again be occupied by more than one particles.

Bosonized hamiltonian. We will consider, for simplicity, bosonized expressions for non-interacting fermion hamiltonians of the type (2.22). The bosonic hamiltonian, for the $N = 2$ system, is

$$H'_B = \sum_{k=0}^{\infty} \mathcal{E}(k)[(\delta_k + \delta_{k-1})\, \delta_{\hat{\nu}} + (\delta_k + \delta_{k-\hat{s}_1-1})\, \delta_{\hat{\nu}-1} + (\delta_{k-\hat{s}_2} + \delta_{k-\hat{s}_1-1})\, \delta_{\hat{\nu}-2}] - \sum_{k=0}^{N-1} \mathcal{E}(k) \tag{5.13}$$

where $\hat{\nu} = \sum_{n=1}^{\infty} b^\dagger_n b_n = N - b^\dagger_0 b_0$. The expression for general N is given in (6.4). Bosonization of (2.28) *a la* the second bosonic system can also be worked out.

6. Applications to $c = 1$ matrix model

In this paper we will only report some preliminary observations on $c = 1$. Detailed results, the full import of which are yet to be understood, will be presented elsewhere.

As emphasized before, our bosonization formulae do not depend on the choice of a specific fermion hamiltonian. It can thus be applied to the $c = 1$ model for which the fermion hamiltonian involves an upside down harmonic oscillator potential:

$$H_F = \sum_n \mathcal{E}(n) \psi^\dagger_n \psi_n = \frac{1}{2} \int dx\ \psi^\dagger(x) \left[-\frac{1}{\beta^2} \frac{d^2}{dx^2} - x^2 + A^2 \right] \psi(x) \tag{6.1}$$

Here $\mathcal{E}(n)$ are the eigenvalues of the hamiltonian $\hat{h} = [-\frac{1}{\beta^2}\frac{d^2}{dx^2} - x^2 + A^2]$ with an infinite wall at $x = \pm A$ [39]. The spectrum $\mathcal{E}(n)$ and the corresponding eigenfunctions can be explicitly evaluated in terms of parabolic cylinder functions. In the scaling limit $N \to \infty, \beta \to \infty, N/\beta \to A^2/(2\pi)$ one can obtain the following WKB estimate for energy levels close to the Fermi surface:

$$\mathcal{E}(N+m) - \mathcal{E}(N) = cm + O(m^2), \qquad c \equiv \frac{\pi}{\beta \log(\sqrt{2\beta}A)} \tag{6.2}$$

For the first system of bosons, we get a bosonized hamiltonian using (2.23). In the scaling limit and for states close to the fermi level in the fermionic description (corresponding to bosonic states involving a small number of low energy bosonic excitations), the hamiltonian (2.23) becomes, after a simple calculation using (6.2):

$$H_B \approx c \sum_{k=1}^{N} \sum_{i=k}^{N} a_i^\dagger a_i + \text{const.} = c \sum_{k=1}^{N} k a_k^\dagger a_k + \text{const.} \tag{6.3}$$

We will shortly remark on the relation of this hamiltonian to that of the known relativistic boson. The hamiltonian in terms of the second system of bosons can also be written down (generalizing (5.13) to arbitrary N):

$$H'_B = \sum_{k=1}^{N} \delta \left(\sum_{n=1}^{\infty} b_n^\dagger b_n - k \right) \sum_{p=1}^{k} \left[\mathcal{E}(\hat{s}_p + N - p) - \mathcal{E}(N-p) \right] . \tag{6.4}$$

where $\hat{s}_p$ is defined in (5.6).

Remarks: 1. *D0 branes:*

(a) The bosonic operators $d_n^\dagger \equiv b_n^\dagger b_0$ correspond to creation operators for D0-branes in two dimensional bosonic string theory (D0 branes in two dimensional string theory were first described in the matrix model language in [40, 41]). This can be verified as follows. In the fermionic language, $d_n^\dagger$ acting on the fermi sea kicks the fermion at the fermi level up by n levels; such an excitation represents a D0 brane at the energy level $N + n$[9]. Note that the number of $d^\dagger$ excitations in any state of the bosonized theory is bounded by N, but the individual energy level of such an excitation is unbounded above (see section 5), as expected of D0 branes.

(b) A localized D0 brane corresponds to an appropriate linear combination of $d_n^\dagger$'s so as to form a single-particle coherent state. Such D0 branes are unstable and the corresponding "tachyon potential" [42] is described by the D0 brane hamiltonian (6.4). The "bottom" of the tachyon potential is, of course, given by the "vacuum" state annihilated by all the $d_n, n = 1, .., \infty$ oscillators (in terms of the $b, b^\dagger$ oscillators this state is $(b_0^\dagger)^N |vac\rangle, b_n|vac\rangle = 0, n = 0, 1, \ldots, \infty$). The energy difference ΔE between the initial energy and the bottom

[9]The occasional statements that D0 branes in 2D string theory correspond to fermions are somewhat loose. Taken literally, they would imply that D0 branes cannot be created in a fixed-N theory, at least before the double scaling limit. A more appropriate picture (emphasized in, e.g., [35]) is that D0 branes are to be understood as fermion-antifermion pairs, since kicking a fermion from the fermi level upwards is such an excitation. Our representation of the D0 brane in terms of the bosonic $d_n, d_m^\dagger$ oscillators is a precise formulation of this idea.

of the potential can be easily computed using (6.4), and for a D0 brane described by $d_k^\dagger$, $\Delta E = \mathcal{E}(N+k) - \mathcal{E}(N)$.

(c) The hamiltonian (6.4) describes a nonperturbative interacting hamiltonian for D0 branes, thus playing the role of a hamiltonian for open string field theory.

(d) The d-oscillators are composites of the b-oscillators and do not have simple Heisenberg commutators. Although $[d_m^\dagger, d_n^\dagger] = [d_m, d_n] = 0$, $[d_n, d_m^\dagger] = b_0^\dagger b_0 \delta_{mn} - b_m^\dagger b_n$. It will be interesting if this represents interesting statistics among D0 branes.

2. *Holes:* The operators $a_i^\dagger$ represent bosonic excitations corresponding to creation of holes (analogues of the LLM giant gravitons). In the finite N theory these are again related to D0 branes, just as for LLM the giant gravitons provide a dual description of the dual giant states (see section 5). It will be interesting, however, to understand these statements in the double scaling limit, where these holes are distinguished from the "particles" and represent some new states [43]. This may provide some nontrivial hints about the double scaling limit as well as the new states.

3. *Tachyons:* It is interesting to understand the double-scaled limit of our bosonization formulae to find the relation of these bosons with the tachyon of two-dimensional string theory. This problem is similar to the problem of connecting the graviton to the a- and the b-oscillators in the LLM case. It turns out that for small fluctuations around the Fermi vacuum, the hamiltonian (6.3) as well as the creation/annihilation operators $(a_i^\dagger, a_j)$ coincide with those for the well-known relativistic [16, 17, 19, 18] bosons and hence get connected to the massless closed string tachyon in the double-scaled limit. Details of this will be presented elsewhere.

7. Discussion

We list below the main results and some comments:

1. We have found exact operator bosonizations of a finite number of fermions which can be described by states in a countable Hilbert space.

2. There are two systems of bosons. In the first system of bosons, the number of bosonic particles is not constrained but each bosonic particle moves in a finite dimensional Hilbert space, the dimensionality being the number of fermionic particles. In the second systems of bosons, the number of bosonic particles has an upper bound which is the number of fermions and the single-particle Hilbert space is the same as that of the fermions.

3. In the LLM example, the first kind of bosons correspond to giant gravitons in the and the second kind of bosons correspond to dual giant gravitons.

4. In the $c = 1$ example, the second kind of bosons correspond to unstable D0 branes. The interpretation of the first kind of bosons, which represent holes in the Fermi sea, is not clear.

5. The finite number of bosonic energy levels in case of the first bosonization implies a fuzzy compact phase space (equivalently, bosons on a lattice). In the LLM case, it has the important implication that finite rank of the boundary SYM theory corresponds to NC geometry in the bulk, where the fundamental quanta describing such geometry are giant gravitons rather than the perturbative gravitons.

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6. The description of $c = 1$ in terms of a finite number of bosonic modes suggests a similar NC structure, although whether it survives in the double-scaled limit, and if so in what form, remains an open question.

7. The system of fermions on a circle, described in section 2, is closely related to the problem of formation of baby universes as described in [44]. It would be interesting to investigate whether in this example also our bosons are related to microscopic gravitational degrees of freedom, as in the LLM case.

It is interesting to speculate whether the NC geometry in the LLM example is a property only of the specific half-BPS sector of type IIB theory or if it is more general. One of the points of emphasis in this paper is that a finite number of modes in the bulk can imply noncommutative gravity. Whether this feature survives in the full theory is an important question that needs further investigation.

One of the intriguing aspects of our exact bosonization (the first system of bosons) is that the symmetries of the fermion system (even the number of spacetime dimensions) are rather intricately hidden in the bosonic theory. Although at first sight this feature may not appear to be particularly welcome, it may hint at a more abstract description of spacetime in which the latter is a derived or emergent concept. In this context, it might be useful to study the application of this bosonization to fermion systems in higher dimensions along the lines briefly outlined in section 2.1.3.

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A. Details of computations: the first bosonization

In this appendix we have put together details of some of the computations summarized in the main text. We will begin by explaining in the first subsection how we arrived at the first bosonization described in section 2. In the second subsection we will give proof of the algebra (2.4) for the oscillators which are defined in (2.15) and (2.16) by their action on arbitrary fermion states. In the third subsection, we will give an expression for the general fermion bilinear, $\psi^\dagger_{n+m}\ \psi_n$, in terms of the bosonic oscillators and indicate some details of its derivation. In the fourth subsection we will prove that this expression satisfies the W_∞ algebra for small values of m for arbitrary N. Finally, in the last subsection we will use the expression for the bilinear for $N = 2$ to prove that it satisfies the W_∞ algebra for arbitrary m.

A.1 Origin of the bosonization formulae

Here we will describe the steps involved in deducing the bosonization formulae given in section 2 from the map (4.2). Consider first the action of oscillator creation operators $a^\dagger_k$

for $k < N$ on a general fermi state. We have,

$$\begin{aligned} a_k^\dagger\ |f_1,\dots,f_N\rangle =&\ a_k^\dagger\ |r_1,\dots,r_k,\dots,r_N\rangle \\ =&\ \sqrt{r_k+1}\ |r_1,\dots,r_k+1,\dots,r_N\rangle. \\ =&\ \sqrt{f_{N-k+1}-f_{N-k}}\ |f_1,\dots,f_{N-k},\ f_{N-k+1}+1,\dots,f_N+1\rangle \end{aligned} \tag{A.1}$$

The first and last equalities above follow from the state map (4.2) and the second follows from the definition of the creation operator. Similarly, for $a_N^\dagger$ we get

$$\begin{aligned} a_N^\dagger\ |f_1,\dots,f_N\rangle =&\ a_N^\dagger\ |r_1,\dots,r_N\rangle \\ =&\ \sqrt{r_N+1}\ |r_1,\dots,r_N+1\rangle. \\ =&\ \sqrt{f_1+1}\ |f_1+1,\dots,f_N+1\rangle \end{aligned} \tag{A.2}$$

These are exactly the expressions given in equation (2.15). Expressions for annihilation operators can be obtained similarly and these coincide with those given in (2.16).

Consider now the fermion bilinear $\psi_m^\dagger\ \psi_n$. Let us first set $m = n$. Acting on a general fermion state, we get

$$\begin{aligned} \psi_n^\dagger\ \psi_n\ |f_1,\dots,f_N\rangle =&\ \sum_{k=1}^N \delta(f_k-n)\ |f_1,\dots,f_N\rangle \\ =&\ \sum_{k=1}^N \delta\Big(\sum_{i=1}^k r_{N-k+i}+k-1-n\Big)\ |r_1,\dots,r_N\rangle \\ =&\ \sum_{k=1}^N \delta\Big(\sum_{i=k}^N a_i^\dagger\ a_i+N-k-n\Big)\ |f_1,\dots,f_N\rangle. \end{aligned} \tag{A.3}$$

The first equality is a simple consequence of the fact that $\psi_n^\dagger\ \psi_n$ kills the state unless one of the fermions is occupying the level n. The second and last equalities then follow from the state map (4.2). The last expression coincides with the first line of (2.21). The bosonized formula for the general bilinear given below in (A.8) can be obtained by similar manipulations of the state map. The calculation is essentially elementary, though longer and more tedious.

A.2 Proof of the oscillator algebra

Let us consider the first of the equations in (2.15). Applying a_l, $l < k$, on both sides of this equation and using (2.16), we get

$$\begin{aligned} a_l a_k^\dagger\ |f_1,\dots,f_N\rangle =&\ \sqrt{f_{N-k+1}-f_{N-k}}\quad a_l\ |f_1,\dots,f_{N-k},f_{N-k+1}+1,\dots,f_N+1\rangle \\ =&\ \sqrt{(f_{N-k+1}-f_{N-k})(f_{N-l+1}-f_{N-l}-1)}\ \times \\ &\ |f_1,\dots,f_{N-k},f_{N-k+1}+1,\dots,f_{N-l}+1,f_{N-l+1},\dots,f_N\rangle. \end{aligned} \tag{A.4}$$

Reversing the order, starting with the first of the equations in (2.16) and applying a creation operator on both sides, we get

$$\begin{aligned} a_k^\dagger a_l\ |f_1,\dots,f_N\rangle =&\ \sqrt{f_{N-l+1}-f_{N-l}-1}\quad a_k^\dagger\ |f_1,\dots,f_{N-l},f_{N-l+1}-1,\dots,f_N-1\rangle \\ =&\ \sqrt{(f_{N-k+1}-f_{N-k})(f_{N-l+1}-f_{N-l}-1)}\ \times \\ &\ |f_1,\dots,f_{N-k},f_{N-k+1}+1,\dots,f_{N-l}+1,f_{N-l+1},\dots,f_N\rangle. \end{aligned} \tag{A.5}$$

The right-hand side of this equation is identical to that of (A.4). It follows that $[a_l, a_k^\dagger] = 0$ for $l < k$. One can similarly prove this for $l > k$. For $l = k$, however, we get

$$\begin{aligned} a_k a_k^\dagger \, |f_1, \ldots, f_N\rangle &= \sqrt{f_{N-k+1} - f_{N-k}} \quad a_k \, |f_1, \ldots, f_{N-k}, f_{N-k+1} + 1, \ldots, f_N + 1\rangle \\ &= (f_{N-k+1} - f_{N-k}) \, |f_1, \ldots, f_N\rangle, \end{aligned} \tag{A.6}$$

and

$$\begin{aligned} a_k^\dagger a_k \, |f_1, \ldots, f_N\rangle &= \sqrt{f_{N-k+1} - f_{N-k} - 1} \quad a_k^\dagger \, |f_1, \ldots, f_{N-k}, f_{N-k+1} - 1, \ldots, f_N - 1\rangle \\ &= (f_{N-k+1} - f_{N-k} - 1) \, |f_1, \ldots, f_N\rangle. \end{aligned} \tag{A.7}$$

If follows that $[a_k, a_k^\dagger] = 1$. Combining with the above result, we see that our bosonic operators satisfy the standard oscillator algebra $[a_l, a_k^\dagger] = \delta_{lk}$.

A.3 Derivation of the bosonized expression for generic fermion bilinear

We will first give the bosonized expression for the fermion bilinear and then indicate the key steps in its derivation. The expression given below is valid only for $m > 0$. The expression for $m < 0$ can be obtained from it by conjugation. We have,

$$\begin{aligned} \psi_{n+m}^\dagger \, \psi_n = \sum_{k=1}^{N-1} \Bigg[& \sigma_k^m \, {\sigma_{k+1}^\dagger}^m \, \theta_+(a_k^\dagger a_k - m) - \sum_{r_k=0}^{\infty} \sigma_{k-1}^{m-1-r_k} \, {\sigma_k^\dagger}^{m-2-r_k} \, \sigma_k^{r_k} \, {\sigma_{k+1}^\dagger}^{r_k+1} \\ & \times \theta_-(a_k^\dagger a_k - m + 1) \, \theta_+(a_{k-1}^\dagger a_{k-1} + a_k^\dagger a_k - m + 1) \, \delta(a_k^\dagger a_k - r_k) \\ & + \sum_{j=2}^{k-1} (-1)^j \sum_{r_{k-j+1}=0}^{\infty} \sum_{r_{k-j+2}=0}^{\infty} \cdots \sum_{r_k=0}^{\infty} \sigma_{k-j}^{m-j-\sum_{i=1}^{j} r_{k-j+i}} \\ & \times {\sigma_{k-j+1}^\dagger}^{m-j-1-\sum_{i=1}^{j} r_{k-j+i}} \, \sigma_{k-j+1}^{r_{k-j+1}} \, {\sigma_{k-j+2}^\dagger}^{r_{k-j+1}} \cdots \sigma_{k-1}^{r_{k-1}} \, {\sigma_k^\dagger}^{r_{k-1}} \\ & \times \sigma_k^{r_k} \, {\sigma_{k+1}^\dagger}^{r_k+1} \, \theta_-\left(\sum_{i=1}^{j} a_{k-j+i}^\dagger \, a_{k-j+i} - m + j \right) \\ & \times \theta_+\left(\sum_{i=0}^{j} a_{k-j+i}^\dagger \, a_{k-j+i} - m + j \right) \Pi_{i=1}^{j} \delta(a_{k-j+i}^\dagger \, a_{k-j+i} - r_{k-j+i}) \\ & + (-1)^k \sum_{r_1=0}^{\infty} \cdots \sum_{r_k=0}^{\infty} {\sigma_1^\dagger}^{m-1-k-\sum_{i=1}^{k} r_i} \, \sigma_1^{r_1} \, {\sigma_2^\dagger}^{r_1} \cdots \sigma_{k-1}^{r_{k-1}} \, {\sigma_k^\dagger}^{r_{k-1}} \\ & \times \sigma_k^{r_k} \, {\sigma_{k+1}^\dagger}^{r_k+1} \, \theta_-\left(\sum_{i=1}^{k} a_i^\dagger a_i - m + k \right) \Pi_{i=1}^{k} \delta(a_i^\dagger a_i - r_i) \Bigg] \\ & \times \delta\left(\sum_{i=k+1}^{N} a_i^\dagger a_i - n + N - k - 1 \right) \\ + {\sigma_1^\dagger}^m \, \delta\left(\sum_{i=1}^{N} a_i^\dagger a_i - n + N - 1 \right). \end{aligned} \tag{A.8}$$

Let us now explain the main steps in the derivation of this expression. Consider the action of the fermion bilinear on a generic state. The result is zero unless the level n is occupied, that is

$$\psi^\dagger_{n+m}\;\psi_n\;|f_1,\ldots,f_N\rangle=\;\sum_{k=1}^{N}\delta_{nf_k}\;\psi^\dagger_{f_1}\;\cdots\;\psi^\dagger_{f_{k-1}}\;\psi^\dagger_{n+m}\;\psi^\dagger_{f_{k+1}}\;\cdots\;\psi^\dagger_{f_N}|0\rangle_F \tag{A.9}$$

Furthermore, the right-hand side above vanishes unless the level $(n+m)$ is unoccupied. Assuming this is the case, we must consider several possibilities, depending on the exact value of m. This is done by rewriting the above equation as follows:

$$\begin{aligned}\psi^\dagger_{n+m}\;\psi_n\;|f_1,\ldots,f_N\rangle=\;&\sum_{k=1}^{N-1}\delta_{nf_k}\;\Bigg[\sum_{l=f_k+1}^{f_{k+1}-1}\delta_{f_k+m,l}\;\psi^\dagger_{f_1}\;\cdots\;\psi^\dagger_{f_{k-1}}\;\psi^\dagger_{l}\;\psi^\dagger_{f_{k+1}}\;\cdots\;\psi^\dagger_{f_N}|0\rangle_F\\&-\sum_{l=f_{k+1}+1}^{f_{k+2}-1}\delta_{f_k+m,l}\;\psi^\dagger_{f_1}\;\cdots\;\psi^\dagger_{f_{k-1}}\;\psi^\dagger_{f_{k+1}}\;\psi^\dagger_{l}\;\psi^\dagger_{f_{k+2}}\cdots\;\psi^\dagger_{f_N}|0\rangle_F\\&+\cdots\cdots\cdots\cdots\cdots\cdots\\&+(-1)^{N-k}\sum_{l=f_N+1}^{\infty}\delta_{f_k+m,l}\;\psi^\dagger_{f_1}\;\cdots\;\psi^\dagger_{f_{k-1}}\;\psi^\dagger_{f_{k+1}}\;\cdots\;\psi^\dagger_{f_N}\;\psi^\dagger_{l}\;|0\rangle_F\Bigg]\\&+\;\delta_{nf_N}\;\psi^\dagger_{f_1}\;\cdots\;\psi^\dagger_{f_{N-1}}\;\psi^\dagger_{f_N+m}\;|0\rangle_F.\end{aligned} \tag{A.10}$$

The first term in the square brackets above corresponds to the possibility that $(n+m)=(f_k+m)$ lies between (f_k+1) and $(f_{k+1}-1)$, the second term to the possibility that it lies between $(f_{k+1}+1)$ and $(f_{k+2}-1)$ and so on. The term outside the square brackets corresponds to $k=N$, that is to the possibility that $n=f_N$. We can write the above equivalently as

$$\begin{aligned}\psi^\dagger_{n+m}\;\psi_n\;|f_1,\ldots,f_N\rangle=\;&\sum_{k=1}^{N-1}\delta_{nf_k}\;\Bigg[\sum_{l=f_k+1}^{f_{k+1}-1}\delta_{f_k+m,l}|\tilde f_1=f_1,\ldots,\tilde f_{k-1}=f_{k-1},\\&\qquad\qquad\tilde f_k=l,\tilde f_{k+1}=f_{k+1},\ldots,\tilde f_N=f_N\rangle\\&-\sum_{l=f_{k+1}+1}^{f_{k+2}-1}\delta_{f_k+m,l}\;|\tilde f_1=f_1,\ldots,\tilde f_{k-1}=f_{k-1},\\&\qquad\qquad\tilde f_k=f_{k+1},\tilde f_{k+1}=l,\tilde f_{k+2}=f_{k+2},\ldots,\tilde f_N=f_N\rangle\\&+\cdots\cdots\cdots\cdots\cdots\cdots\\&+(-1)^{N-k}\sum_{l=f_N+1}^{\infty}\delta_{f_k+m,l}\;|\tilde f_1=f_1,\ldots,\tilde f_{k-1}=f_{k-1},\\&\qquad\qquad\tilde f_k=f_{k+1},\tilde f_{k+1}=f_{k+2},\ldots,\tilde f_{N-1}=f_N,\tilde f_N=l\rangle\Bigg]\\&+\;\delta_{nf_N}\;|\tilde f_1=f_1,\ldots,\tilde f_{k-1}=f_{k-1},\tilde f_k=f_{k+1},\\&\qquad\tilde f_{k+1}=f_{k+2},\ldots,\tilde f_{N-1}=f_N,\tilde f_N=f_N+m\rangle.\end{aligned} \tag{A.11}$$

Using the state map (4.2), the right-hand side can be re-expressed in terms of the bosonic oscillator states and the oscillator numbers that refer to them. We get,

$$
\begin{aligned}
&\psi^\dagger_{n+m}\,\psi_n\,|f_1,\ldots,f_N\rangle \\
&= \sum_{k=1}^{N-1}\delta\biggl(\sum_{i=1}^{k} r_{N-k+i}-n+k-1\biggr)\biggl[\theta_+(r_{N-k}-m)\times \\
&\qquad\qquad |\tilde r_1=r_1,\ldots,\tilde r_{N-k-1}=r_{N-k-1},\ \tilde r_{N-k}=r_{N-k}-m, \\
&\qquad\qquad\qquad\qquad \tilde r_{N-k+1}=r_{N-k+1}+m,\tilde r_{N-k+2}=r_{N-k+2}\ \cdots\ ,\tilde r_N=r_N\rangle \\
&\quad -\theta_-(r_{N-k}-m+1)\ \theta_+(r_{N-k-1}+r_{N-k}-m+1)\times \\
&\qquad\quad |\tilde r_1=r_1,\ldots,\tilde r_{N-k-2}=r_{N-k-2},\tilde r_{N-k-1}=r_{N-k-1}+r_{N-k}-m+1, \\
&\qquad\qquad \tilde r_{N-k}=m-2-r_{N-k},\tilde r_{N-k+1}=r_{N-k+1}+r_{N-k}+1, \\
&\qquad\qquad\quad \tilde r_{N-k+2}=r_{N-k+2}\ \cdots\ ,\tilde r_N=r_N\rangle \\
&\quad +\cdots\cdots\cdots\cdots\cdots\cdots \\
&\quad +(-1)^{N-k}\theta_-\biggl(\sum_{i=1}^{N-k} r_i-m+N-k\biggr)\times \\
&\qquad\quad |\tilde r_1=\sum_{i=1}^{N-k} r_i-m+N-k,\ \tilde r_2=r_1,\ldots,\tilde r_{N-k}=r_{N-k-1}, \\
&\qquad\qquad \tilde r_{N-k+1}=r_{N-k+1}+r_{N-k}+1,\tilde r_{N-k+2}=r_{N-k+2},\ldots,\tilde r_N=r_N\rangle\biggr] \\
&\quad +\delta\biggl(\sum_{i=1}^{N} r_i-n+N-1\biggr)\ |\tilde r_1=r_1+m,\ \tilde r_2=r_2,\ldots,\tilde r_N=r_N\rangle.
\end{aligned} \tag{A.12}
$$

Now, using the bosonic creation and annihilation operators it is easy to re-express every bosonic state appearing on the right-hand side above in terms of the state $|r_1,\ldots,r_N\rangle$ to which the fermionic state $|f_1,\ldots,f_N\rangle$ corresponds under the state map (4.2). This results in the bosonized operator expression for the fermi bilinear which is given in (A.8).

A.4 Proof of W_∞ algebra for small values of m for arbitrary N

Consider the bilinears $\psi^\dagger_{n+1}\,\psi_n$ and $\psi^\dagger_{n+2}\,\psi_{n+1}$. From the W_∞ algebra (B.6), we get

$$[\psi^\dagger_{n+2}\ \psi_{n+1},\psi^\dagger_{n+1}\ \psi_n]=\psi^\dagger_{n+2}\ \psi_n. \tag{A.13}$$

It is clear from this equation that we can generate bosonized expressions for $\psi^\dagger_{n+m}\,\psi_n$ for any m merely from the knowledge of bosonized expression for $\psi^\dagger_{n+1}\,\psi_n$ by using the W_∞ algebra. However, here we will use the expressions given in (2.21) for $m=0,\ 1$ and 2, which are special cases of (A.8), to compute the commutator and verify that the result agrees with the right-hand side.

To compute the commutator we will need to use the following identities, in addition to those given in (2.9):

$$g(a^\dagger_k a_k)\ \sigma^\dagger_k=\sigma^\dagger_k\ g(a^\dagger_k a_k+1),\qquad g(a^\dagger_k a_k)\ \sigma_k=\sigma_k\ g(a^\dagger_k a_k-1)\ \theta_+(a^\dagger_k a_k-1), \tag{A.14}$$

where g is any function of the number operator. We are now ready to do the computation using the second of (2.21) in the commutator. All terms in the commutator involve products of two delta-functions and several vanish because these are incompatible. The surviving terms are

$$\begin{aligned}
&[\psi^\dagger_{n+2}\ \psi_{n+1}, \psi^\dagger_{n+1}\ \psi_n] \\
&\quad = \sigma^\dagger_1\ \delta\Big(\sum_{i=1}^{N} a^\dagger_i a_i - n + N - 2\Big)\ \sigma^\dagger_1\ \delta\Big(\sum_{i=1}^{N} a^\dagger_i a_i - n + N - 1\Big) \\
&\quad +\sigma^\dagger_1\ \delta\Big(\sum_{i=1}^{N} a^\dagger_i a_i - n + N - 2\Big)\ \sigma_1\ \sigma^\dagger_2\ \theta_+(a^\dagger_1 a_1 - 1)\ \delta\Big(\sum_{i=2}^{N} a^\dagger_i a_i - n + N - 2\Big) \\
&\quad -\sigma_1\ \sigma^\dagger_2\ \theta_+(a^\dagger_1 a_1 - 1)\ \delta\Big(\sum_{i=2}^{N} a^\dagger_i a_i - n + N - 2\Big)\ \sigma^\dagger_1\ \delta\Big(\sum_{i=1}^{N} a^\dagger_i a_i - n + N - 2\Big) \\
&\quad + \sum_{k=1}^{N-1} \sigma_k\ \sigma^\dagger_{k+1}\ \theta_+(a^\dagger_k a_k - 1)\ \delta\Big(\sum_{i=k+1}^{N} a^\dagger_i a_i - n + N - k - 2\Big) \\
&\quad \times \sigma_k\ \sigma^\dagger_{k+1}\ \theta_+(a^\dagger_k a_k - 1)\ \delta\Big(\sum_{i=k+1}^{N} a^\dagger_i a_i - n + N - k - 1\Big) \\
&\quad - \sum_{k=2}^{N-1} \sigma_k\ \sigma^\dagger_{k+1}\ \theta_+(a^\dagger_k a_k - 1)\ \delta\Big(\sum_{i=k+1}^{N} a^\dagger_i a_i - n + N - k - 1\Big) \\
&\quad \times \sigma_{k-1}\ \sigma^\dagger_k\ \theta_+(a^\dagger_{k-1} a_{k-1} - 1)\ \delta\Big(\sum_{i=k}^{N} a^\dagger_i a_i - n + N - k - 1\Big)
\end{aligned} \tag{A.15}$$

The first term comes from the commutator of the first term in the bosonized expression for the bilinear. The next two terms come from the cross-commutator between the first term and $k = 1$ piece of the second term (which involves sum over k). The last two terms are from the commutator of the second term; this survives only when the same k is picked from the two sums or if the k's differ by 1. Further simplification requires the use of the relations (A.14), (2.9) and $\theta_+(a^\dagger a - 1) = \theta_+(a^\dagger a) - \delta(a^\dagger a) = 1 - \delta(a^\dagger a)$. The result is precisely the expression on the right-hand side of the of the last of (2.21).

Another test of the W_∞ algebra comes from the use of the bosonized expression of the fermion bilinear conjugate to $\psi^\dagger_{n+1}\ \psi_n$. We have,

$$\begin{aligned}
\psi^\dagger_n\ \psi_{n+1} = \ \Big(\psi^\dagger_{n+1}\ \psi_n\Big)^\dagger &= \sigma_1\ \delta\Big(\sum_{i=1}^{N} a^\dagger_i a_i - n + N - 2\Big) \\
&+ \sum_{k=1}^{N-1} \sigma_{k+1}\ \sigma^\dagger_k\ \delta\Big(\sum_{i=k+1}^{N} a^\dagger_i a_i - n + N - k - 2\Big).
\end{aligned} \tag{A.16}$$

From the W_∞ algebra we see that

$$[\psi^\dagger_{n+1}\ \psi_n, \psi^\dagger_l\ \psi_{l+1}] = \delta_{nl}(\psi^\dagger_{n+1}\ \psi_{n+1} - \psi^\dagger_n\ \psi_n). \tag{A.17}$$

Using the bosonized expressions in the commutator, we get

$$
\begin{aligned}
&[\psi^\dagger_{n+1}\ \psi_n, \psi^\dagger_l\ \psi_{l+1}] \\
&\quad = \delta_{nl}\Bigg\{\theta_+(a_1^\dagger a_1 - 1)\ \delta\Bigg(\sum_{i=1}^N a_i^\dagger a_i - n + N - 2\Bigg) - \delta\Bigg(\sum_{i=1}^N a_i^\dagger a_i - n + N - 1\Bigg) \\
&\quad + \sum_{k=1}^{N-1}\Bigg[\theta_+(a_{k+1}^\dagger a_{k+1} - 1)\ \delta\Bigg(\sum_{i=k+1}^N a_i^\dagger a_i - n + N - k - 2\Bigg) \\
&\quad - \theta_+(a_k^\dagger a_k - 1)\ \delta\Bigg(\sum_{i=k+1}^N a_i^\dagger a_i - n + N - k - 1\Bigg)\Bigg]\Bigg\}
\end{aligned} \tag{A.18}
$$

Now, let us replace $\theta_+(a^\dagger a - 1)$ by the equivalent expression $(1 - \delta(a^\dagger a))$ in all the three places. All the terms containing double delta-functions mutually cancel, except the one coming from $k = (N-1)$ of the first term in square brackets. But this has two incompatible delta-functions and so vanishes. The result for the right-hand side is

$$
\delta_{nl}\sum_{k=1}^N\Bigg[\delta\Bigg(\sum_{i=1}^N a_i^\dagger a_i - n + N - k - 1\Bigg) - \delta\Bigg(\sum_{i=1}^N a_i^\dagger a_i - n + N - k\Bigg)\Bigg], \tag{A.19}
$$

which is precisely the bosonized expression one gets by using the first of (2.21) in the the right-hand side of (A.17). We see that for this test to work out, delicate cancellations between various terms were required.

A.5 Proof of W_∞ algebra for all m for $N = 2$

For $N = 2$, bosonized expressions for the bilinear for all values of m have been given in (2.20). This is the first nontrivial, yet calculationally manageable case. We have checked that in this case the W_∞ algebra is satisfied. Here we will indicate the main steps in the calculation. Like in the above calculations, delicate cancellations between various terms in the commutator are required for the algebra to work out, as we shall see.

We will be interested in the commutator

$$
[\psi^\dagger_{n+m}\ \psi_n, \psi^\dagger_{l+p}\ \psi_l] = \delta_{l+p,n}\ \psi^\dagger_{l+p+m}\ \psi_l - \delta_{n+m,l}\ \psi^\dagger_{n+m+p}\ \psi_n, \tag{A.20}
$$

for $m \geq p > 0$. Other cases can be treated similarly.

There are three terms in the bosonized expression for the fermion bilinear given in (2.20). It is easy to see that the self-commutator of the first two terms reproduces the first two terms required by the bosonized expression for the right-hand side of (A.20). To prove that the bosonized expressions satisfy the (A.20), we then need to show that the self-commutator of the third term, together with all the cross-commutator terms, reproduces the required third term on the right-hand side. The self-commutator of the third term works out to be

$$
\begin{aligned}
&\sigma_1^{\dagger m-p}\ \sigma_2^{\dagger p}\ \theta_+(p + l - n - 1)\ \delta(a_1^\dagger a_1 - n + l + 1)\ \delta(a_2^\dagger a_2 - l) \\
&- \sigma_1^{m-p}\ \sigma_2^{\dagger m}\ \theta_+(p + l - m - n - 1)\ \theta_+(m + n - l - 1) \\
&\times \delta(a_1^\dagger a_1 - l + n + 1)\ \delta(a_2^\dagger a_2 - n),
\end{aligned} \tag{A.21}
$$

while the cross-commutator of the first two terms gives

$$\begin{aligned}
&\sigma_1^{m-p}\ \sigma_2^{\dagger m}\ [\theta_+(p+l-m-n-1)-\theta_+(l-n-m-1)]\\
&\times\delta(a_1^\dagger a_1-l+n+1)\ \delta(a_2^\dagger a_2-n)\\
&-\sigma_1^{\dagger m-p}\ \sigma_2^{\dagger p}\ \theta_-(n-p-l-1)\ \delta(a_1^\dagger a_1-n+l+1)\ \delta(a_2^\dagger a_2-l).
\end{aligned}\tag{A.22}$$

The contributions in (A.21) and (A.22) are very similar and can be combined. Thus the first term of (A.21) can be combined with the second term of (A.22) using $\theta_+(p+l-n-1)-\theta_-(n-p-l-1)=-\delta_{p+l,n}$. Simplifying in this way gives the net combined contribution

$$\begin{aligned}
&-\delta_{p+l,n}\ \sigma_1^{\dagger m-p}\ \sigma_2^{\dagger p}\ \delta(a_1^\dagger a_1-p+1)\ \delta(a_2^\dagger a_2-l)\\
&+\delta_{m+n,l}\ \sigma_1^{m-p}\ \sigma_2^{\dagger m}\ \delta(a_1^\dagger a_1-m+1)\ \delta(a_2^\dagger a_2-n),
\end{aligned}\tag{A.23}$$

Finally, the cross-commutator between the first two terms and the third term gives

$$\begin{aligned}
&\Bigg[-\delta_{p+l,n}\sum_{r_1=0}^{p-2}\sigma_1^{\dagger m+p-2-r_1}\ \ \sigma_1^{r_1}\ \sigma_2^{\dagger r_1+1}\ \delta(a_1^\dagger a_1-r_1)\ \delta(a_2^\dagger a_2-l)\\
&+\delta_{m+n,l}\sum_{r_1=0}^{p-2}\sigma_1^{\dagger p-2-r_1}\ \ \sigma_1^{r_1+m}\ \sigma_2^{\dagger r_1+m+1}\ \delta(a_1^\dagger a_1-r_1-m)\ \delta(a_2^\dagger a_2-n)\Bigg]\\
&-\Bigg[m\leftrightarrow p, n\leftrightarrow l\Bigg].
\end{aligned}\tag{A.24}$$

Notice that by changing the summation variable from r_1 to (r_1+m) in the second term in the first square brackets, we get a summand that is similar to the first term, but with a summation range for the new variable from m to $(m+p-2)$. This nicely combines with the first term in the second square brackets to give an overall summation range for r_1 from 0 to $(m+p-2)$, as is required if W_∞ algebra is to be satisfied by the bosonized expressions. To be precise, the summation range in the first term in the second square brackets is from 0 to $(m-2)$ only, so the contribution for $r_1=(m-1)$ is missing from the extended summation range. Fortunately, the terms in (A.23) precisely supply this missing contribution. Taking this into account and simplifying, we find that the net result of the commutator calculation is

$$\begin{aligned}
&-\delta_{p+l,n}\sum_{r_1=0}^{m+p-2}\sigma_1^{\dagger m+p-2-r_1}\ \ \sigma_1^{r_1}\ \sigma_2^{\dagger r_1+1}\ \delta(a_1^\dagger a_1-r_1)\ \delta(a_2^\dagger a_2-l)\\
&+\delta_{m+n,l}\sum_{r_1=0}^{m+p-2}\sigma_1^{\dagger m+p-2-r_1}\ \ \sigma_1^{r_1}\ \sigma_2^{\dagger r_1+1}\ \delta(a_1^\dagger a_1-r_1)\ \delta(a_2^\dagger a_2-n).
\end{aligned}\tag{A.25}$$

This is exactly the third term in the bosonized form of the right-hand side of (A.20). Hence we have proved that our bosonization satisfies the W_∞ algebra for $N=2$.

B. Quantum phase space distributions and star products

Some assorted references carrying more detailed versions of formulae in this appendix are [45, 28, 29, 46, 30, 47, 48, 34, 49].

B.1 Single particle

Consider the (infinite-dimensional) Hilbert space of a particle in one dimension, carrying a representation of the Heisenberg algebra $[\hat{x}, \hat{p}] = i\hbar$. The Wigner phase space distribution of the particle in a wavefunction $|\psi\rangle$ is given by

$$W(q,p) \equiv \int d\eta\ e^{-ip\eta/\hbar}\ \psi^\dagger(q-\eta/2)\ \psi(q+\eta/2) = \langle\psi|\hat{g}_{q,p}|\psi\rangle,\ \hat{g}_{q,p} \equiv e^{i\frac{q\hat{p}+p\hat{x}}{\hbar}} \tag{B.1}$$

There are other choices for the phase space distribution, e.g. the Husimi distribution, given by

$$H(z,\bar{z}) \equiv |\langle z|\psi\rangle|^2 = \int \frac{dq'dp'}{2\pi\hbar} W(q',p') e^{-\frac{1}{2\hbar}[(q'-q)^2+(p'-p)^2]},\ z \equiv q+ip \tag{B.2}$$

Using Wigner phase space distribution one can define a map between operators and functions, with an associated star product:

$$\begin{aligned}
&\hat{O} \mapsto O_w(q,p) \equiv \mathrm{Tr}(\hat{O}g_{q,p}), \qquad \langle\psi|\hat{O}|\psi\rangle = \int dqdp O_w(q,p) W(q,p)\\
&\hat{A}\hat{B} \mapsto A_w(q,p) *_w B_w(q,p) \equiv e^{i\hbar[\partial_p\partial_{q'} - \partial_{p'}\partial_q]}(A_w(q,p)B_w(q',p'))|q'=q, p'=p
\end{aligned} \tag{B.3}$$

The inverse map $O_w \to \hat{O}$ corresponds to Weyl operator ordering of the function O_w. The above star product $*_w$ is called the Moyal star product.

The corresponding definitions for Husimi distribution are

$$\begin{aligned}
&\hat{O} \mapsto O_h(z,\bar{z}) \equiv \langle z|\hat{O}|z\rangle, \qquad \langle\psi|\hat{O}|\psi\rangle = \int dqdp O_h(q,p) H(q,p)\\
&\hat{A}\hat{B} \mapsto A_h(z,\bar{z}) *_h B_h(z,\bar{z}) \equiv e^{i\hbar[\partial_z\partial_{\bar{z}'} - \partial_{z'}\partial_{\bar{z}}]}(A_w(z,\bar{z})B_w(z',\bar{z}'))|z'=z
\end{aligned} \tag{B.4}$$

where $*_h$ is called the Voros star product.

Example of the harmonic oscillator (3.1):

The Wigner and Husimi distributions for the wavefunction $|j\rangle, j = 0, 1, \ldots, \infty$, are

$$\begin{aligned}
W_j(q,p) &= \frac{(-1)^j}{\pi\hbar} e^{-\frac{q^2+p^2}{\hbar}} L_j(\frac{2(p^2+q^2)}{\hbar})\\
H_j(z,\bar{z}) &= \frac{1}{2\pi\hbar(j!)} e^{-|z|^2}|z|^{2j}
\end{aligned} \tag{B.5}$$

B.2 Second quantization

Fermions: Consider a system of N fermions, as in section 2.

The operators $\Phi_{mn} = \psi^\dagger_m\psi_n$ (see eq. (2.3))) satisfy the W_∞ algebra [DMW]

$$[\Phi_{mn}, \Phi_{m'n'}] = \delta_{m'n}\Phi_{mn'} - \delta_{mn'}\Phi_{m'n} \tag{B.6}$$

Φ_{mn} are the basic operators in any one dimensional fermion field theory in a given fermion number sector. A basis free notation is

$$\Phi = \sum_{m,n} \Phi_{mn}|n\rangle\langle m| \tag{B.7}$$

The second quantized Wigner phase space density $\hat{W}_F(q,p)$ is a linear combination of Φ_{mn}

$$\hat{W}_F(q,p) = Tr(\Phi g_{q,p}) = \int d\eta\ e^{-ip\eta/\hbar}\ \psi^\dagger(q-\eta/2)\ \psi(q+\eta/2) \tag{B.8}$$

The expectation value of $\hat{W}_F(q,p)$ in the fermi state (2.2) is the sum of the single-particle distributions, $\sum_m W_{f_m}(q,p)$. The second quantized Husimi phase space density $\hat{H}_F(q,p)$ is given by

$$\hat{H}_F(z,\bar{z}) = Tr(\Phi|z\rangle\langle z|) = \sum_{m,n} \psi_m^\dagger \psi_n (\chi_m(z))^* \chi_n(z), \quad \chi_n(z) = \langle z|n\rangle \tag{B.9}$$

Bosons: The second quantized phase space distributions for bosons are given by similar formulas,

$$\begin{aligned}
\hat{W}_B(q,p) &= Tr(\Phi_B g_{q,p}) = \int d\eta\ e^{-ip\eta/\hbar}\ \phi^\dagger(q-\eta/2)\ \phi(q+\eta/2),\ \Phi_B = \phi_i^\dagger \phi_j |i\rangle\langle j| \\
\hat{H}_B(z,\bar{z}) &= Tr(\Phi|z\rangle\langle z|) = \sum_{i,j} \phi_i^\dagger \phi_j (\chi_i(z))^* \chi_j(z), \quad \chi_i(z) = \langle z|i\rangle
\end{aligned} \tag{B.10}$$

C. New bosonic oscillator representation of U(K)

We begin by noting that the W_∞ algebra (B.6), generated by $\psi_m^\dagger \psi_n, m,n = 0,1,\ldots\infty$, has the following nested subalgebras

$$\mathrm{U}(1) \subset \mathrm{U}(2) \subset \mathrm{U}(3) \cdots \subset W_\infty \tag{C.1}$$

where the subalgebra $\mathrm{U}(K), K = 1,\ldots,\infty$ is generated by the finite $\psi_m^\dagger \psi_n,\ m,n = 0,1,\ldots,K-1$. The structure constants in (B.6) are easily seen to be the structure constants of U(K).

The representation of the subalgebra U(K), provided by $\mathcal{F}_N^K$, defined as the Hilbert space of N fermions in the first K levels $m = 0,1,\ldots,K-1$, is the rank-N antisymmetric tensor representation (dimension ${}^K C_N$).

We will bosonize $\mathcal{F}_N^K$ and its operators, using (A.8) and its special cases and in the process will obtain novel[10] bosonic representations of U(K).

We will start with the simplest examples of small N.

C.1 The $N = 1$ example

Here $\mathcal{F}_N^K = \mathcal{H}^K$, the single-particle Hilbert space of fermions, truncated to the first K levels. We rewrite the equations of (2.19) involving the first K of fermionic oscillators. For $m,n = 0,1,\ldots,K-1$,

$$\begin{aligned}
\psi_m^\dagger \psi_m &= P_m \equiv \delta(a^\dagger a - m) \\
\psi_m^\dagger \psi_n &= (\sigma^\dagger)^{m-n} P_n,\ m > n \\
\psi_m^\dagger \psi_n &= P_m (\sigma)^{n-m},\ m < n
\end{aligned} \tag{C.2}$$

[10]different from Schwinger representations where the generators are bilinears in bosonic oscillators.

Here $P_m = |m\rangle\langle m|$, the projection operator. $a, a^\dagger$ denote $a_1, a_1^\dagger$ and $\sigma, \sigma^\dagger$ are defined as in (2.8).

Now, although $a, a^\dagger$, and consequently $\sigma, \sigma^\dagger$, are infinite dimensional matrices (Heisenberg algebra can only have infinite dimensional representations), the operators on the r.h.s. of (C.2) have the matrix form (in the basis $\frac{(a^\dagger)^m}{\sqrt{m!}}|0\rangle$)

$$\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \tag{C.3}$$

where A is a $K \times K$ matrix.

Since the operator map (2.19) ensures that algebra of fermion bilinears is reproduced by the bosonic expressions, the right hand side of (C.2) provides a bosonic representation of U(K). We will consider some explicit, small K, examples below.

The case $K = 2$: bosonic representation of U(2) or SU(2). For $K = 2$ Φ_{mn} generate the U(2) algebra. The bosonic versions of the generators are $P_0, P_1, a^\dagger P_0, P_0 a$. These correspond to matrices of the form (C.3), where A is a 2×2 matrix, assuming the following values, respectively

$$P_0 \to \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad P_1 \to \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad a^\dagger P_0 \to \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad P_0 a \to \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{C.4}$$

These provide a bosonic construction of the spin-1/2 representation of SU(2) ($P_0 + P_1$ represents the trace part of U(2) algebra).

$K = 3$: bosonic representation of U(3) or SU(3). (C.2) now gives the following bosonic generators of U(3): $P_0, P_1, P_2, a^\dagger P_0, \frac{1}{\sqrt{2}}(a^\dagger)P_1, (a^\dagger)^2 P_0, P_0 a, \frac{1}{\sqrt{2}} P_1 a, P_0 a^2$. $P_0 + P_1 + P_2$ represents the trace part and the rest provide the fundamental representation **3** of SU(3). The matrix representations are of the form (C.3) with A equal to standard 3×3 SU(3) matrices.

For general K ($N = 1$) we get a bosonic construction of the fundamental (dim K) representation of SU(K).

C.2 The $N = 2$ example

The relevant bosonization formulae are (2.20). The bosonic Hilbert space is a linear combination of states $|mn\rangle = \frac{(a_1^\dagger)^m (a_2^\dagger)^n}{\sqrt{m!n!}}|00\rangle$. Let us define projectors $P_{mn} = |mn\rangle\langle mn| = \delta(a_1^\dagger a_1 - m)\delta(a_2^\dagger a_2 - n)$.

We will start with examples of small K. The first non-trivial case is $K = 3$, for which (2.20) gives

$$\psi_0^\dagger \psi_0 = \sum_{m=0}^{\infty} P_{m0}, \psi_1^\dagger \psi_1 = P_{00} + \sum_{m=1}^{\infty} P_{m1}, \psi_1^\dagger \psi_1 = P_{01} + P_{10} + \sum_{m=0}^{\infty} P_{m2}$$

$$\psi_1^\dagger \psi_0 = \sigma_1 \sigma_2^\dagger \sum_{m=0}^{\infty} P_{m0}, \psi_2^\dagger \psi_0 = (\sigma_1 \sigma_2^\dagger)^2 \sum_{m=0}^{\infty} P_{2m} - \sigma_2^\dagger P_{00}, \psi_2^\dagger \psi_1 = \sigma_1^\dagger P_{00} + \sigma_1 \sigma_2^\dagger \sum_{m=1}^{\infty} P_{m1}$$

The bosonic operators are infinite dimensional matrices, but are of a triangular form (cf. (C.3))

$$\begin{pmatrix} A & A' \\ 0 & A'' \end{pmatrix} \tag{C.5}$$

where A is a 3×3 matrix, corresponding to the subspace $\mathcal{H}_3 = Span\{|00\rangle, |01\rangle, |10\rangle\}$. The matrices A can be worked out and they correspond to an irrep. of SU(3) (viz. the representation $\bar{\mathbf{3}}$).

C.3 Bosonization of N fermions in a K-level system

We will now give the result for general N, K which is straightforward to derive:

- The bosonization formulae (A.8) can be applied to bosonize N fermions in a K-level system.
- The bosonization formulae (A.8) give a novel bosonic construction of the general rank-N antisymmetric tensor rep of SU(K) in terms of N bosonic oscillators.

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